2.5 LU Matrix Factorization

Matrices can be factored (called matrix decomposition) in several different ways. One common method is LU factorization, or Lower-Upper factorization. The matrix $A = \begin{pmatrix} 3 & 4 & -2 \\ 6 & 10 & -3 \\ -3 & 2 & 8 \end{pmatrix}$ can be factored as $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ where $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} = L$ (lower matrix) and $\begin{pmatrix} 3 & 4 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} = U$ (upper matrix).

Example 1 Verify $L\,U = A$.

Using LU to Solve the equation $A\,\mathbf{x} = \mathbf{b}$

LU factorization is used to solve the equation $A\,\mathbf{x} = \mathbf{b}$, especially when this equation needs to be solved for multiple $\mathbf{b}$, $A\,\mathbf{x} = \mathbf{b}_1$, $A\,\mathbf{x} = \mathbf{b}_2$, etc.

Consider the equation $A\,\mathbf{x} = \mathbf{b}$, and the factorization $A = L\,U$:

$$A\,\mathbf{x} = \mathbf{b}$$
$$(L\,U)\,\mathbf{x} = \mathbf{b}$$
$$L\,(U\,\mathbf{x}) = \mathbf{b}$$

We can solve this by letting $\mathbf{y} = U\,\mathbf{x}$, giving $L\,\mathbf{y} = \mathbf{b}$; first solve $L\,\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\,\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

Example 2 Solve the equation $\begin{pmatrix} 3 & 4 & -2 \\ 6 & 10 & -3 \\ -3 & 2 & 8 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ 11 \\ 41 \end{pmatrix}$ using the LU decomposition from above.

LU Factorization Algorithm

Suppose we can use elementary matrices, E_p, E_{p-1}, …, E_2, E_1, (**without row swaps**) to reduce A to echelon form, i.e., $E_p \ldots E_1 A = U$ (the upper matrix.) Then, since elementary matrices are invertible, $A = (E_p \ldots E_1)^{-1} U$, meaning $(E_p \ldots E_1)^{-1} = L$, the lower matrix. And, the same elementary matrices will also reduce L to the identity matrix I, or $E_p \ldots E_1 L = I_n$ (why?) Therefore, $L = (E_p \ldots E_1)^{-1} I_n$. This gives a method for finding L and U.

Algorithm for an LU Factorization

1. Reduce A to echelon form U, using elementary operations if possible (no row swaps).
2. Place entries in L such that the sequence of row operations reduces L to I_n.

Example 3 Find the elementary matrices E_1, E_2, and E_3 that reduces A to U, and show that $L = (E_3 E_2 E_1)^{-1}$.

Example 4 Use the LU factorization for $A = \begin{pmatrix} 2 & 4 & -3 \\ 4 & 5 & 1 \\ 2 & -5 & 32 \end{pmatrix}$ to solve the equation $A\mathbf{x} = \begin{pmatrix} 13 \\ 18 \\ 3 \end{pmatrix}$.