2.3 Characterizations of Invertible Matrices

Theorem 8 The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is invertible.

Example 1 Demonstrate (or verify) all the properties in Theorem 8 for the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ -2 & 1 & -3 \end{pmatrix}$.

Example 2 Is matrix A invertible? $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 2 & 4 \\ 2 & 0 & -2 \end{pmatrix}$ Use Theorem 8.

Invertible Transformations

Recall matrix multiplication is a composition of linear transformations, i.e., $T:\mathbb{R}^n \to \mathbb{R}^m$. Suppose, however, T is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$. The transformation is said to be invertible if there exists a transformation $S:\mathbb{R}^n \to \mathbb{R}^n$ such that

1. $S(T(\boldsymbol{x})) = \boldsymbol{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and
2. $T(S(\boldsymbol{x})) = \boldsymbol{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Example 3 Let the transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ be $T(\boldsymbol{x}) = \begin{pmatrix} 1 & 5 \\ 2 & 11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Find the image of $\boldsymbol{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ under T, that is $\boldsymbol{u} = T(\boldsymbol{v})$, and find an inverse transformation S, such that $S(\boldsymbol{u}) = \boldsymbol{v}$.

Theorem 9

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be a transformation matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\boldsymbol{x}) = A^{-1}\boldsymbol{x}$ is the unique function satisfying equations 1 and 2 above.

Ill-Conditioned Matrices

It's possible to have an invertible matrix that is "nearly-singular", meaning a small change will make it singular. Such a matrix can give very different results due to round-off errors. The **condition number** for a square matrix can indicate how ill-conditioned a matrix is; the identity matrix has a condition number of 1, and a singular matrix has an infinite condition number.

Example 4 Solve the system of equations: $\begin{cases} 1.51\,x_1 + 2.27\,x_2 = b_1 \\ 1.13\,x_1 + 1.70\,x_2 = b_2 \end{cases}$ for $\mathbf{b} = \begin{pmatrix} 7.105 \\ 5.319 \end{pmatrix}$, and then with $\mathbf{b}$ rounded to two and one decimal places, using $\mathbf{x} = A^{-1}\,\mathbf{b}$. Also, use the 2×2 inverse formula to find A^{-1}.