2.2 The Inverse of a Matrix

Recall the *multiplicative inverse property*: $5 \cdot \frac{1}{5} = 5 \cdot 5^{-1} = 5^{-1} \cdot 5 = 1$. A similar property is true for **square** matrices.

We say an $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that $AC = CA = I_n$. We call C the **inverse of *A*** and denote it A^{-1}. Note, for scalars we often write 5^{-1} as $\frac{1}{5}$, but for matrices, there is no such thing as division, so it is *always* written A^{-1}. A matrix that does **not** have an inverse is often called a **singular** matrix, and an invertible matrix is called a **non-singular** matrix.

Example 1 Show that the matrix $C = \begin{pmatrix} -11 & 9 \\ 5 & -4 \end{pmatrix}$ is the inverse for $A = \begin{pmatrix} 4 & 9 \\ 5 & 11 \end{pmatrix}$

Theorem 4 Inverse of a 2×2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $ad - bc = 0$ then A is not invertible. (Show this is always true).

Example 2 Find the inverse for $A = \begin{pmatrix} 5 & 2 \\ 7 & 8 \end{pmatrix}$

Theorem 5

If A is an $n \times n$ invertible matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 3 Use inverse matrices to solve the system:

$$\begin{aligned} 2x - 5y &= 3 \\ x + 4y &= 5 \end{aligned}$$

Theorem 6 Properties of Inverses

Let A and B be $n \times n$ invertible matrices.

a. $(A^{-1})^{-1} = A$
b. Since A and B are both invertible, then so is AB. In fact, $(AB)^{-1} = B^{-1}A^{-1}$.
c. Since A is invertible, so is A^T. In fact, $(A^T)^{-1} = (A^{-1})^T$.

Elementary Matrices

Consider the following matrices:

$E_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ $E_2=\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}$ $E_3=\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}$ and $A=\begin{pmatrix}a1&a2&a3\\b1&b2&b3\\c1&c2&c3\end{pmatrix}$

💡 What effect does each elementary matrix, E_i above, have when multiplying A, i.e., $E_1 A$ etc?

Now, consider the elementary matrices that could be used to row reduce the matrix $\begin{pmatrix}3&2\\7&5\end{pmatrix}$ into the identity matrix $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

$$\begin{pmatrix}\frac{1}{3}&0\\0&1\end{pmatrix}\begin{pmatrix}3&2\\7&5\end{pmatrix}=\begin{pmatrix}1&\frac{2}{3}\\7&5\end{pmatrix}\Longrightarrow\begin{pmatrix}1&0\\-7&1\end{pmatrix}\begin{pmatrix}1&\frac{2}{3}\\7&5\end{pmatrix}=\begin{pmatrix}1&\frac{2}{3}\\0&\frac{1}{3}\end{pmatrix}\Longrightarrow\begin{pmatrix}1&0\\0&3\end{pmatrix}\begin{pmatrix}1&\frac{2}{3}\\0&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}1&\frac{2}{3}\\0&1\end{pmatrix}\Longrightarrow\begin{pmatrix}1&\frac{-2}{3}\\0&1\end{pmatrix}\begin{pmatrix}1&\frac{2}{3}\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

We can also apply these all in one step: $E_4E_3E_2E_1A=\begin{pmatrix}1&\frac{-2}{3}\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&3\end{pmatrix}\begin{pmatrix}1&0\\-7&1\end{pmatrix}\begin{pmatrix}\frac{1}{3}&0\\0&1\end{pmatrix}\begin{pmatrix}3&2\\7&5\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, which means the product $E_4E_3E_2E_1$ is the inverse of A!! 😮 (Also, since elementary row operations are reversible, each elementary matrix has an inverse.) Furthermore, if we multiply $E_4E_3E_2E_1$ to the identity matrix we get the inverse matrix! This gives us a method to find the inverse of a matrix.

Theorem 7

An $n\times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations reducing A to I_n also transforms I_n to A^{-1}.

Algorithm For Finding A^{-1}

Row reduce the augmented matrix $[A\ \ I]$. If A is row equivalent to I, then $[A\ \ I]$ is row equivalent to $[I\ \ A^{-1}]$. Otherwise, A does not have an inverse.

Example 4 Use row reduction to find inverse of the matrix $\begin{pmatrix}3&1\\2&5\end{pmatrix}$.

Example 5 Use row reduction to find the inverse of $\begin{pmatrix}1&-1&2\\1&2&0\\-2&1&-3\end{pmatrix}$.