# 1.9 The Matrix of a Linear Transformation

In this section we want to try and determine a particular transformation matrix knowing only the transformation on two vectors.

**Example 1** Suppose $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $T$ be a linear transformation such that

$$T(\mathbf{e}_1) = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \text{ and } T(\mathbf{e}_2) = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$$

Find a formula for the image of an arbitrary $\mathbf{x} \in \mathbb{R}^2$. (Note the $[\mathbf{e}_1 \;\; \mathbf{e}_2]$ is the $I_2$ identity matrix.)

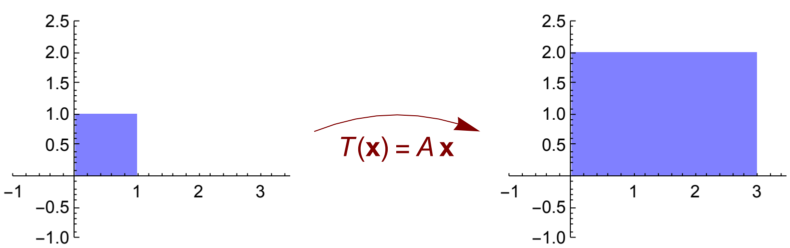
### Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $A$ such that

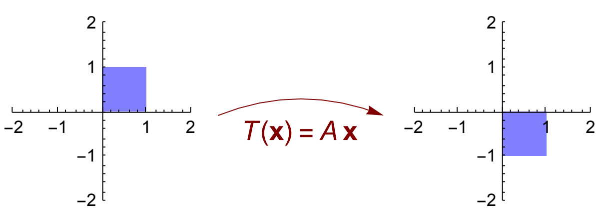
$$T(\mathbf{x}) = A\,\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Furthermore $A = [T(\mathbf{e}_1) \;\; \cdots \;\; T(\mathbf{e}_n)]$, where $\mathbf{e}_j$ is the $j$th column of an identity matrix in $\mathbb{R}^n$. $A$ is call the **standard matrix for the linear transformation** $T$.

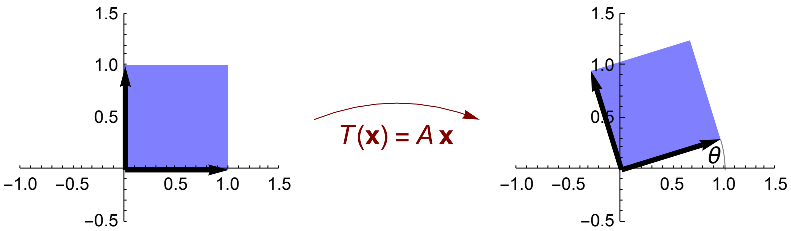
**Example 2** Find the matrix that expands any vector $\mathbf{x}$ in $\mathbb{R}^2$ by 3 horizontally and by 2 vertically. That is, transforms the point (1, 1) to (3, 2), or the vector $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ to $\begin{pmatrix} 12 \\ 10 \end{pmatrix}$, e.g. $T\left[\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right] = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$. Geometrically, the transformation of the unit square:




**Example 3** Find a matrix for the reflection over the horizontal axis.

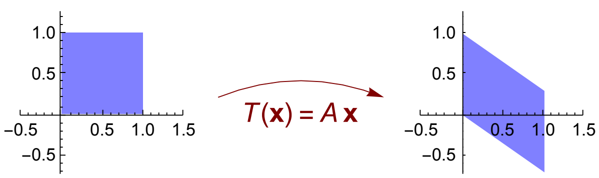


**Example 4** Find a rotation matrix for a counterclockwise rotation of $\theta$. Think about where $\mathbf{e}_1$ and $\mathbf{e}_2$ are mapped to.




**Example 5** Analyze why the transformation $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, is a horizontal shear. Again, think about the transformation on $\mathbf{e}_1$ and $\mathbf{e}_2$.

**Example 6** Find a matrix that will shear vertically down by a factor of $k$. Geometrically:




**Example 7** Find the transformation matrix that scales the $x$ coordinate of a point in $\mathbb{R}^2$ by 2, reflects across the y-axis, and rotates 60°.

## Theorem 11

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

## Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:

a. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$;
b. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.