1.8 Intro to Linear Transformations

Example 1 Show that T_1 maps or transforms $\mathbf{x} \in \mathbb{R}^4$ in to $\mathbb{R}^2$, T_2 maps $\mathbf{y} \in \mathbb{R}^2$ in to $\mathbb{R}^3$:

a. $T_1(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 1 & 3 & -1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}$

b. $T_2(\mathbf{y}) = B\mathbf{y}$ where $B = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Example 2 Is there more than one $\mathbf{y}$ that gives the same image as $B\mathbf{y}$ above?

Example 3 Is it possible to have the vector $\mathbf{c} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ be in the transformation of T_2?

Domain, Codomain, Range of a Transformation

Suppose we have a transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, that is, each vector $\mathbf{x}$ in $\mathbb{R}^n$ is assigned to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, $\mathbb{R}^m$ is called the **codomain**, and the set of all images $T(\boldsymbol{x})$ is called the **range** of T.

Example 4 Make a sketch of a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ where the range is a plane.

Matrix Transformations

Example 5 Let $A = \begin{pmatrix} 2 & 1 \\ -3 & 1 \\ 1 & 2 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 7 \\ 3 \\ 8 \end{pmatrix}$, and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$ and define the linear transformation $T(\boldsymbol{x}) = A\,\boldsymbol{x}$.

a) Find $T(\boldsymbol{u})$, the image of $\mathbf{u}$ under the transformation T.

b) Find an $\boldsymbol{x}$ in $\mathbb{R}^2$ whose image under T is $\boldsymbol{b}$. (Is there more than one $\mathbf{x}$?)

c) Determine if $\boldsymbol{c}$ is in the range of the transformation of T.

Example 6 Analyze the geometrical transformation $T(\mathbf{x}) = A\,\mathbf{x}$ of $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ on a vector $\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Linear Transformations

Recall a linear operator has the properties: 1) $A(\mathbf{u} + \mathbf{v}) = A\,\mathbf{u} + A\,\mathbf{v}$, and 2) $A(c\,\mathbf{u}) = c\,A\,\mathbf{u}$. Written in transformation function notation we have...

Definition

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T;

(ii) $T(c\,\mathbf{u}) = c\,T(\mathbf{u})$ for all $\mathbf{u}$ and all scalars c.

Will a linear transformation always transform the zero vector to the zero vector, e.g, $T(\mathbf{0}) = \mathbf{0}$?

Example 7 Define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\boldsymbol{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Find the images under T of $\boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{u} + \boldsymbol{v}$.