1.7 Linear Independence

In previous sections we asked the question as to whether the equation $A\mathbf{x} = \mathbf{0}$ had a solution besides the trivial $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. To expand and clarify this idea, we have several definitions and theorems.

Definition

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, c_2, \cdots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Example 1 Determine if the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ are linearly independent. If possible, find a linear dependence relation for $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

Linear Independence of Matrix Columns

The column vectors of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Example 2 Determine if the columns of the matrix $A = \begin{pmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{pmatrix}$ are linearly independent.

Example 3 Determine if each set of vectors below are linearly independent.

a. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

b. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Theorem 7 Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. This does not mean that *every* vector in S is a linearly combination of the others.

Example 4 Determine if the columns of the matrix $A = \begin{pmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{pmatrix}$ form a linearly independent set.

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Example 5 For matrix A above, show that $\mathbf{v}_3$ is a linear combination of the other vectors, but $\mathbf{v}_4$ is **not** a linear combination of the other vectors.

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set of vectors is linearly dependent.

Example 6 If a set S of vectors contains the zero vector, what can be said about the linear independence of the set of vectors?