1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Another way to write a system of equations and linear combinations of vectors is by way of matrix and vector multiplication.

Definition

If A is an $m \times n$ matrix with columns $\mathbf{a}_1 \cdots \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, the product of A and $\mathbf{x}$, denoted $A\mathbf{x}$, is the *linear combination of the columns of* A *using the entries of* $\mathbf{x}$ *as the weights*; that is

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Example 1 Given $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$, evaluate $[\mathbf{a}_1 \quad \mathbf{a}_2]\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Example 2 Evaluate $\begin{pmatrix} 2 & 5 & 1 \\ 1 & 3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Example 3 Using matrix A and $\mathbf{x}$ in example 2, solve $A\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Theorem 3

If A is an $m \times n$ matrix with columns $\mathbf{a}_1 \cdots \mathbf{a}_n$, and if $\mathbf{x}$ and $\mathbf{b}$ are in $\mathbb{R}^n$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$, as well as the system of linear equations whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]$.

Note: the definition of $A\mathbf{x}$ along with Theorem 3 means the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A. This means the following questions are equivalent:

1) is $\mathbf{b}$ in span$\{a_{1,} \ldots, a_n\}$?
2) is $A\mathbf{x} = \mathbf{b}$ consistent?

Example 4 Let $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 7 & -3 \\ 1 & -1 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for **all** b_1, b_2, b_3?

Solution: Reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

So far we have developed several equivalent statements (many more to follow):

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular matrix A, either they are all true statements or they are all false:

a. For each **b** in $\mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.
b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Note: A is assumed to be a coefficient matrix, **not** an augmented matrix.

Another Way to Multiply Matrices

Example 5 Multiply the following matrices:

a) $\begin{pmatrix} 2 & 3 \\ -3 & 4 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

b) $\begin{pmatrix} -2 & 4 & 6 \\ 3 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$

Theorem 5

If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$;
b. $A(c\boldsymbol{u}) = c(A\boldsymbol{u})$