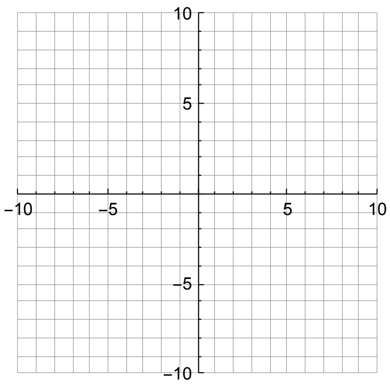
1.3 Vector Equations

Vectors

A matrix that is $m \times 1$ is called a column vector: $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \{3, 5, 6\} = (3, 5, 6) = \langle 3, 5, 6 \rangle$. These are all equivalent vectors in $\mathbb{R}^3$ using different notation. Vectors are **added** component-wise, e.g., $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$. Vectors can also be multiplied by a scalar, e.g., $3\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ -15 \end{pmatrix}$. In $\mathbb{R}^2$ the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ can be represented as the directed line segment from the origin (0, 0) to the point (a, b).

Example 1 Graph the vectors $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\mathbf{u} + \mathbf{v}$, and $\boldsymbol{w} = 3\,\boldsymbol{u} + 2\,\boldsymbol{v}$.



Algebraic Properties of Vectors in $\mathbb{R}^n$

For all **u**, **v**, and **w** in $\mathbb{R}^n$, and scalars c and d:

i)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	v)	$c(\mathbf{u} + \mathbf{v}) = c\,\mathbf{u} + c\,\mathbf{v}$
ii)	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	vi)	$(c + d)\,\mathbf{u} = c\,\mathbf{u} + d\,\mathbf{u}$
iii)	$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	vii)	$c(d\,\mathbf{u}) = (c\,d)\,\mathbf{u}$
iv)	$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$	viii)	$1\,\mathbf{u} = \mathbf{u}$

Linear Combinations of Vectors

Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \ldots \boldsymbol{v}_p$ in $\mathbb{R}^n$, and scalars $c_1, c_2, c_3 \ldots c_p$, the vector **w** given by

$$\mathbf{w} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3 + \cdots + c_p\,\mathbf{v}_p$$

is called a **linear combination** of the vectors $\mathbf{v}_1 \ldots \mathbf{v}_p$ with the weights $c_1 \ldots c_p$. Example 1 is a linear combination.

Example 2 Given $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 2 \\ 14 \end{pmatrix}$, rewrite $x_1\,\boldsymbol{v}_1 + x_2\,\boldsymbol{v}_2 = \boldsymbol{b}$, and solve for x_1 and x_2.

Solving a Vector Equation

A vector equation

$$x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \cdots x_n \boldsymbol{a}_n = \boldsymbol{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n \quad \boldsymbol{b}]$$

The vector $\boldsymbol{b}$ can be generated by a linear combination of $\boldsymbol{a}_1, \ldots \boldsymbol{a}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix.

Example 3 Find the linear combination of the vectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$ that gives the following vectors:

a) $\mathbf{b}_1 = \begin{pmatrix} -8 \\ 19 \\ 1 \end{pmatrix}$

b) $\mathbf{b}_2 = \begin{pmatrix} 5 \\ 9 \\ 0 \end{pmatrix}$

Definition

If $\mathbf{v}_1, \ldots \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the set of *all* linear combinations of $\mathbf{v}_1, \ldots \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or generated) by $\mathbf{v}_1, \ldots \mathbf{v}_p$. That is, $\text{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is the set of all vectors that can be written in the form $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \cdots + c_p \mathbf{v}_p$.

Example 4 What can be said about the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, and $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ from Example 3? Explain geometrically.