1.2 Row Reduction and Echelon Form

Definition

A rectangular matrix is in **echelon form** if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following two conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Echelon Form Versus Reduced Echelon Form

For the first matrix in **echelon form**, the leading entries ■ can be any nonzero value, and the * values can be any value. The second matrix is in reduced echelon form.

$$\begin{pmatrix} 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

Theorem 1

Each matrix is row equivalent to one and only one reduced echelon matrix. (This can be proved after section 4.3)

***Mathematica* Code**

```
A = RowReduce[{{1, -7, 0, 6, 5}, {0, 0, 1, -2, -3}, {-1, 7, -4, 2, 7}}] // MatrixForm
```

$$\begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition

For a matrix A the **pivot position** corresponds to the position of the leading 1's in the reduced echelon form matrix, and the **pivot column** is the column the pivot position is in.

Row Reduction Algorithm

Forward Phase

Step 1 Begin with the left most nonzero column as the first pivot column. The pivot position is at the top.

Step 2 If necessary, interchange rows so that a nonzero entry is in the pivot position.

Step 3 Use elementary row operations to create zero entries in all positions below the pivot.

Step 4 Ignore the previous pivot row and apply steps 1, 2, and 3 to the submatrix that remains. Repeat until there are no more nonzero rows to modify.

Backward Phase

Step 5 for Reduced Echelon Form Start with the rightmost pivot, make it a 1 by a scaling operation, and create zeros above the pivot. Repeat for all pivots.

Example 1 Reduce the matrix $\begin{pmatrix} 1 & -7 & 1 & 4 & 2 \\ 0 & 0 & -2 & 4 & 6 \\ -1 & 7 & -4 & 2 & 7 \end{pmatrix}$ into reduced echelon form.

Solution of Linear Systems

Definition

In a reduced echelon form matrix, each variable in a pivot column is a **basic variable**, and each variable in non-pivot column is a **free** variable. Whenever a consistent system has a free variable, there are infinite solutions depending on any chosen value for each free variable.

Example 2 Find the **general solution** to the system in example 1, and find three particular solutions to the system.

Example 3 For the system of equations with augmented matrix: $\begin{pmatrix} 2 & 1 & 7 & -1 & 6 & -7 \\ 1 & 0 & 2 & 1 & -1 & -1 \\ 1 & 1 & 5 & -1 & 5 & 4 \end{pmatrix}$,

a) Write the matrix in reduced echelon form.
b) Identify the pivot positions, basic variables, and free variables.
c) Write the general solution to the system.
d) Find three specific solutions to the system.

Example 4 Describe *existence* and *uniqueness* in terms of pivot positions and free variables for an $m \times n$ matrix. (**Theorem 2**)