Math 204 In-Class Notes

1.1 Systems of Linear Equations

Solutions of a System of Linear Equations

Recall that a linear equation is an equation in the form: $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$. An example of a *system* of linear equations is:

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + 6x_2 + 2x_3 = 8 \\ -x_1 + 4x_2 + x_3 = 4 \end{cases} \quad \text{or in matrix form} \quad \begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 6 & 2 & 8 \\ -1 & 4 & 1 & 4 \end{pmatrix}$$

Also, a general fact of a system of linear equations (proven later) is:

A system of *linear equations* has either
- (1) No solution (Inconsistent)
- (2) Exactly one solution (consistent)
- (3) Infinitely many solutions (consistent and dependent)

The matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ -1 & 4 & 1 \end{pmatrix}$ is called a *coefficient matrix*, and the matrix $\begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 6 & 2 & 8 \\ -1 & 4 & 1 & 4 \end{pmatrix}$ is called an *augmented* matrix. The **size** of the augmented matrix is 3×4; (m rows) × (n columns).

Solving a Linear System

Row-equivalent matrices are matrices that have an identical solution set. The following are all row-equivalent

$$\begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 6 & 2 & 8 \\ -1 & 4 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

In general, to solve a linear system we can use the *elimination method* (from algebra) to rewrite the matrix into a row-equivalent matrix that is easier to solve.

Example 1 Use the elimination method to solve the system of equations above.

Elementary Row Operations

1. **Replacement:** Replace one row by the sum of itself and a multiple of another row. (i.e., $r2 + 4 \cdot r1 \to r2$)
2. **Interchange:** Swap two rows ($r1 \longleftrightarrow r3$)
3. **Scaling:** Multiply all entries in a row by a nonzero constant ($5 \cdot r2 \to r2$)

Example 2 Use elementary row operations to solve the system of linear equations : $\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ x_1 + 3x_2 + x_3 = 10 \\ 3x_2 + 2x_3 = 22 \end{cases}$

Mathematica Commands

Existence and Uniqueness

Two fundamental questions for any linear system:

1. Does at least one solution ***exist***, (i.e., is it consistent)?
2. If a solution exists, is it the only solution, (is it unique)?

Example 3 Determine if the following system is consistent or inconsistent, dependent or independent:

$$\begin{cases} x_1 + 3x_2 + 4x_4 = 5 \\ 2x_1 + 7x_2 + 2x_3 + 7x_4 = 18 \\ x_2 + 2x_3 - x_4 = 3 \end{cases}$$

Example 4 Find the values of h such that the matrix is an augmented matrix of a consistent system.

$$\begin{pmatrix} 1 & 4 & -7 \\ 3 & h & 2 \end{pmatrix}$$