13.4 Motion in Space: Velocity and, Acceleration

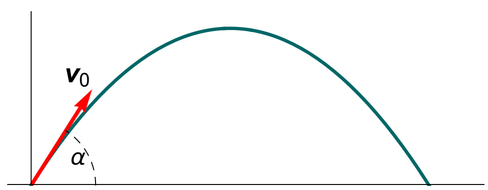
Velocity and Acceleration

If $\boldsymbol{r}(t)$ represents the position function of a particle in space, then $\boldsymbol{r}'(t)$ is the velocity and $\|r'(t)\|$ represents the speed of the particle. Similarly, the acceleration of the particle is $\boldsymbol{r}''(t) = \boldsymbol{v}'(t) = \boldsymbol{a}(t)$.

Example 1 A force $\boldsymbol{F}(t) = 3(2-t)\,\boldsymbol{i} + 6t\,\boldsymbol{j} + 3\,\boldsymbol{k}$ is acting on a 3 kg object. Find its path if it has an initial position of (20, 5, 2) and an initial velocity of $\boldsymbol{v} = \langle 8, 0, 0\rangle$. Recall $\boldsymbol{F} = m\,\boldsymbol{a}$.

Projectile Motion

Consider a projectile shot at an angle of α from the origin with an initial speed of $v_0 = \|\boldsymbol{v}(0)\|$.



Since $\boldsymbol{F} = m\,\boldsymbol{a}$, and the only force acting on the projectile is gravity, we have $\boldsymbol{F} = m\,\boldsymbol{a} = -m\,g\,\boldsymbol{j}$, thus $\boldsymbol{a} = -g\,\boldsymbol{j}$. Integrating gives

$$\boldsymbol{v}(t) = -g\,t\,\boldsymbol{j} + \boldsymbol{v}_0.$$

Integrating again gives

$$\boldsymbol{r}(t) = -\tfrac{1}{2}g\,t^2\,\boldsymbol{j} + \boldsymbol{v}_0\,t + \boldsymbol{r}_0$$

Since $\boldsymbol{r}_0 = \boldsymbol{0}$ and $\boldsymbol{v}_0 = v_0\cos(\alpha)\,\boldsymbol{i} + v_0\sin(\alpha)\,\boldsymbol{j}$, we get

$$\boldsymbol{r}(t) = -\tfrac{1}{2}g\,t^2\,\boldsymbol{j} + [v_0\cos(\alpha)\,\boldsymbol{i} + v_0\sin(\alpha)\,\boldsymbol{j}]\,t + \boldsymbol{0}$$
$$\boldsymbol{r}(t) = v_0\cos(\alpha)\,t\boldsymbol{i} + \left(v_0\sin(\alpha)\,t - \tfrac{1}{2}g\,t^2\right)\boldsymbol{j}$$

Example 2 A projectile is shot with an initial velocity of 80 m/s and angle of elevation 40°, from a position 15 meters above ground level. Where does the projectile hit the ground, and with what speed does it hit the ground (assuming no wind resistance)?

Tangential and Normal Components of Acceleration

It's often useful to write the acceleration vector in terms of **T** and **N**. Since $\boldsymbol{T} = \frac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|} = \frac{\boldsymbol{v}}{v}$, we have $\boldsymbol{v} = v\,\boldsymbol{T}$, and hence

$$\boldsymbol{a} = \boldsymbol{v}' = v'\,\boldsymbol{T} + v\boldsymbol{T}' \qquad (1)$$

Also, $\kappa = \frac{\|\boldsymbol{T}'\|}{\|\boldsymbol{r}'\|} = \frac{\|\boldsymbol{T}'\|}{v}$, so $\|\boldsymbol{T}'\| = \kappa\,v$. The unit normal is defined as $\boldsymbol{N} = \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|}$, which gives $\boldsymbol{T}' = \boldsymbol{N}\,\|\boldsymbol{T}'\| = \boldsymbol{N}\,\kappa\,v$, so equation (1) becomes

$$\boldsymbol{a} = v'\,\boldsymbol{T} + \kappa\,v^2\,\boldsymbol{N}$$

We can also write the tangential component $a_{\boldsymbol{T}} = v'$ and the normal component $a_{\boldsymbol{N}} = \kappa\,v^2$ in terms of $\boldsymbol{r}$,

$$a_{\boldsymbol{T}} = \frac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{\|\boldsymbol{r}'(t)\|} \quad \text{and} \quad a_{\boldsymbol{N}} = \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|}$$

Example 3 Find the tangential and normal components of acceleration for the particle with position function $r(t) = \langle t^2, t^3, 3\,t^2\rangle$.