13.3 Arc Length - Curvature - TNB Vectors

Arc Length

Recall the *arc length* formula for a parametric function in $\mathbb{R}^2$ is $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$. This can be extended to find the length of a vector function in $\mathbb{R}^3$:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$
$$= \int_a^b \| \boldsymbol{r}'(t) \|\, dt$$

Example 1 Find the arc length of the helix $\boldsymbol{r}(t) = 3\cos(t)\,\boldsymbol{i} + 3\sin(t)\,\boldsymbol{j} + \frac{t}{5}\boldsymbol{k}$ from $t = 0$ and $t = 4\pi$.

An arc length function to calculate the length of a curve on the interval [a, t] is given by

$$s(t) = \int_a^t \| \boldsymbol{r}'(u) \|\, du$$

Example 2 Find the arc length function for example (1) for the interval [0, t].

Solving for t we get $t = \frac{5s}{\sqrt{226}}$, which means we can reparametrize $\boldsymbol{r}(t)$ using arc length:

$$\boldsymbol{r}(s) = 3\cos\left(\frac{5s}{\sqrt{226}}\right)\boldsymbol{i} + 3\sin\left(\frac{5s}{\sqrt{226}}\right)\boldsymbol{j} + \frac{s}{\sqrt{226}}\boldsymbol{k}$$

Curvature

The ***curvature*** of a curve $\mathbf{r}(t)$ compares the ***rate*** the unit tangent vector changes with respect to the change in arc length:

$$\kappa = \left\| \frac{d\boldsymbol{T}}{ds} \right\|$$

An easier way to compute this is:

$$\kappa = \frac{\| \boldsymbol{T}'(t) \|}{\| \boldsymbol{r}'(t) \|}$$

This still requires finding the unit vector $\boldsymbol{T}$. An even easier method using only $\boldsymbol{r}(t)$ is

$$K = \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}$$

Example 3 Find the curvature of the twisted cubic $\boldsymbol{r}(t) = \langle t, t^2, t^3\rangle$ when $t = 1$

Example 4 For the case when the function is a plane curve $y = f(x)$, show that

$$K = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$$

and find the curvature of $f(x) = x^2$ when $x = 1$.

Osculating Circle

An osculating circle (kissing circle) is a tangent circle that best approximates a curve at a given point. The radius of the osculating circle is $r = \dfrac{1}{K}$.

Example 5 Find the radius and center of the osculating circle for the function in example (4).

The Unit Normal and Binormal Vectors

Along with the unit tangent vector $\boldsymbol{T}$,two other unit vectors are the unit normal, $\boldsymbol{N}$, and the unit binormal, $\boldsymbol{B}$. These three vectors form an orthogonal *basis* (actually an orthonormal basis) at **any** point along a smooth curve, similar to the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ fixed at the origin:

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|} \quad \text{and} \quad \boldsymbol{B}(t) = \boldsymbol{T}(t)\times\boldsymbol{N}(t) \quad \text{where} \quad \boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$$

Example 6 Find the *trihedral* **TNB** for the twisted cubic from example (3) at the point (1, 1, 1).

Example 7 **Torsion** is the tendency of a curve to twist (think of a roller coaster) and is given by $\tau(t) = -\mathbf{N}(t)\cdot\boldsymbol{B}'(t)$, or $\tau = \dfrac{\|\boldsymbol{r}'\cdot(\mathbf{r}''\times\mathbf{r}''')\|}{\|\mathbf{r}'\times\mathbf{r}''\|}$. Find the torsion at (1, 1, 1) for the twisted cubic.