12.5 Equations of Lines and Planes

Vector Equation of a Line

The equation of a line through the point (x_0, y_0, z_0) in the direction of the vector $\boldsymbol{v} = \langle a, b, c\rangle$ is given by

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\,\boldsymbol{v}$$

where $\boldsymbol{r_0}$ is the initial position vector $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, and $\boldsymbol{r}$ is the position vector for any point on the line.

Example 1 A line contains the point (4, 3, 5) and direction vector $\mathbf{v} = \langle 1, 2, -2\rangle$. Write the equation of the line; find three other points on the line; plot the points and line.

Parametric Form of a Line in $\mathbb{R}^3$

Since, $(x, y, z) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$, the *parametric form* for the equation of a line is:

$$x = x_0 + a\,t \qquad y = y_0 + b\,t \qquad z = z_0 + c\,t$$

Symmetric Form of a line in $\mathbb{R}^3$

Solving the above equations for t we get the *symmetric form* of the equation of a line:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Example 2 Find the parametric and symmetric form for the line passing through the points (4, −2, 1) and (2, 8, −3), and find the point where the line intersects the x-z-plane.

Example 3 Find the distance between the line $\boldsymbol{r} = \langle 3, 2, 8\rangle + t\langle -2, 1, 3\rangle$ and the point (4, −2, −4).

Example 4 Find the distance between the lines. (First show that they are skew and non-intersecting.)

L_1: $x = 2t - 3$ $\quad y = -3t + 4$ $\quad z = t + 2$

L_2: $x = 3s - 1$ $\quad y = 4s + 1$ $\quad z = 2s - 5$

Equations of Planes

The equation of a plane in $\mathbb{R}^3$ can be written in linear from as: $ax + by + cz = d$, (e.g., $2x + 3y + 4z = 12$). This equation can also be written as $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ for any point (x_0, y_0, z_0) on the plane. This is called the *scalar equation* of a plane.

Example 5 Verify the point (−2, 4, 1) is on the plane above and rewrite the equation of the plane using this point.

If we let $\boldsymbol{n} = \langle a, b, c\rangle$, $\boldsymbol{r}_0 = \langle x_0, y_0, z_0\rangle$, and $\boldsymbol{r} = \langle x, y, z\rangle$, the equation of a plane can be written as

$$\boldsymbol{n}\cdot(\boldsymbol{r} - \boldsymbol{r}_0) = 0 \quad \text{or} \quad \boldsymbol{n}\cdot\boldsymbol{r} = \boldsymbol{n}\cdot\boldsymbol{r}_0$$

These are called the vector equations of the plane. Also, the vector $\boldsymbol{n} = \langle a, b, c\rangle$ is the *normal vector* of the plane which is a vector orthogonal to the plane.

Example 6 Find the equation of the plane containing the line $\frac{x-3}{4} = \frac{y+2}{-3} = \frac{z-4}{2}$ and the point (−2, 1, 3).

Example 7 Find the distance from the plane $3x + 5y - 2z = 8$ to the point (2, 2, 5).

Example 8 Find the angle between the planes $2x + 5y + 3z = 8$ and $4x - y + z = 2$.