12.4 The Cross Product

The cross product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is given by

$$\boldsymbol{u} \times \boldsymbol{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - v_1 u_2 \rangle$$

To see where this comes from, we need to look at the *determinant* of a matrix.

The Determinant of a Matrix

Example 1 Find the determinant of the matrix: $\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}$

Example 2 Find the determinant of the matrix: $\begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix}$ Remember: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

The Cross Product Using the Determinant

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

Example 3 Find the cross product of the vectors $\boldsymbol{u} = 2\,\boldsymbol{i} - 3\,\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j} - 2\,\boldsymbol{k}$.

Example 4 Show that the vector $\boldsymbol{u} \times \boldsymbol{v}$ in Ex. 3 is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

Theorem

If θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ then: $\|\boldsymbol{u}\times\boldsymbol{v}\| = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\sin(\theta)\quad (0 \le \theta \le \pi)$

Example 5 Find the area of the parallelogram defined by the vectors $\boldsymbol{u} = 8\,\boldsymbol{i} + 3\,\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = 2\,\boldsymbol{i} + 6\,\boldsymbol{j} - 2\,\boldsymbol{k}$.

Properties of Vector Multiplication

1. $\boldsymbol{u}\times\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{u}$
2. $(c\,\boldsymbol{u})\times\boldsymbol{v} = c(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{u}\times(c\,\boldsymbol{v})$
3. $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\times\boldsymbol{v} + \boldsymbol{u}\times\boldsymbol{w}$
4. $(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{w} + \boldsymbol{v}\times\boldsymbol{w}$
5. $\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}$
6. $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{u}\cdot\boldsymbol{w})\,\boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{v})\,\boldsymbol{w}$

Property 5 is called the **scalar triple product** and can be calculated as a determinant:

$$\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Show that absolute value of the scalar triple product is the volume of a parallelepiped with edges defined by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

Example 6 Find the volume of the parallelepiped with adjacent edges defined by the points (2, −1, 3), (4, 0, 2), (−2, 3, 1), and (3, 2, −2).

Torque

Torque is a measure of the tendency of a body to rotate about the origin, e.g., turning a bolt with a wrench.

The **torque vector** is defined as $\boldsymbol{\tau} = \boldsymbol{r}\times\boldsymbol{F}$ and the magnitude of the torque is $\|\boldsymbol{\tau}\| = \|\boldsymbol{r}\times\boldsymbol{F}\| = \|\boldsymbol{r}\|\,\|\boldsymbol{F}\|\sin(\theta)$.

Example 7 A torque of at least 10 joules is required to tighten a bolt. If a 25 centimeter length wrench has a force of 50 newton applied to it that makes an angle of 70° to the wrench, will this provide enough torque to tighten the bolt?