12.3 The Dot Product

The Dot Product

The **dot product** (or **scalar** product) of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 1 Find the dot product of the vectors $\langle 2, -3, -4 \rangle$ and $\langle 8, 3, -2 \rangle$.

Properties of the Dot Product

If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\,\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Geometry of the Dot Product

If θ is the angle between vectors **a** and **b**, then: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\ \|\mathbf{b}\| \cos(\theta)$

Therefore, the angle between two vectors **a** and **b** can be found: $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\ \|\mathbf{b}\|}$

Example 2 Draw the vectors $\mathbf{a} = 4\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{b} = -3\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}$ and find the angle separating the vectors.

Example 3 Find c such that the vector $\langle 4, -2, c \rangle$ is *orthogonal* to the vector $\langle 3, 1, 2 \rangle$, and write each vector as a unit vector.

Direction Angles

Let α, β, and γ be the angles between a vector $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. Find these angles.

Vector Projections

Scalar projection: The scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is:

$$\text{comp}_{\boldsymbol{v}}\, \boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|}$$

Vector projection: The vector projection $\boldsymbol{u}$ onto $\boldsymbol{v}$ is:

$$\text{proj}_{\boldsymbol{v}}\, \boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\, \boldsymbol{v} \quad \text{or} \quad \text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\, \mathbf{v}$$

Example 4 Find the scalar and vector projection of $\boldsymbol{u} = \langle 3, 6 \rangle$ onto $\boldsymbol{v} = \langle 8, 2 \rangle$, and draw a picture of the projections.

Work

Recall *work* is the amount of force in the direction of motion (or displacement). If a force **F** moves an object along the vector **D**, the amount of force from **F** acting on the object is $\text{comp}_{\mathbf{D}}\, \mathbf{F}$. Multiplying by the displacement $\|\mathbf{D}\|$ we get the amount of work done:

$$\begin{aligned} \text{work} &= \text{comp}_{\mathbf{D}}\, \mathbf{F}\ \|\mathbf{D}\| \\ &= \frac{\mathbf{F} \cdot \mathbf{D}}{\|\mathbf{D}\|} \|\mathbf{D}\| \\ &= \mathbf{F} \cdot \mathbf{D} \end{aligned}$$

Example 5 Calculate the work done in moving a point mass from (2, 1) to (10, 3), in meters, with a 20-Newton force in the direction $\langle 1, 3 \rangle$.