12.2 Vectors

A vector is a quantity that has two characteristics: (1) magnitude, (2) direction, and is often represented as an arrow. Two vectors are equal if they have the same direction and magnitude.

Example 1 A vector with initial point $P_1(3, 2)$ and terminal point $P_2(-4, 7)$ is written in component form, or standard form,

$$\begin{aligned} P_2 - P_1 &= (-4, 7) - (3, 2) \\ &= (-7, 5) \\ \mathbf{v} &= \langle -7, 5 \rangle \end{aligned}$$

Magnitude of a Vector: Norm of $\boldsymbol{v}$

The length, or magnitude, of a vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is denoted as $\|\mathbf{v}\|$ or $|\mathbf{v}|$,

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \quad \text{in } \mathbb{R}^2$$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad \text{in } \mathbb{R}^3$$

Direction Angle of a vector

$$\tan(\theta) = \frac{v_2}{v_1}$$

Example 2 Find the magnitude and direction of the vector: $\langle 20, -50 \rangle$.

Example 3 Find a vector with direction angle $\theta = 65°$ and magnitude 200 in $\mathbb{R}^2$.

Scalar Multiplication

A vector $\boldsymbol{v} = \langle v_1, v_2 \rangle$ can be multiplied by a scalar c

$$\begin{aligned} c\,\mathbf{v} &= c\,\langle v_1, v_2 \rangle \\ &= \langle c\,v_1, c\,v_2 \rangle \end{aligned}$$

Unit Vector: a vector, $\boldsymbol{u}$, of length 1 in the direction $\boldsymbol{v}$ is given by

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Example 4 Find a unit vector in the direction $\langle 3, -8 \rangle$.

Example 5 Find a vector with magnitude 20 in the direction $\langle -3, 2 \rangle$.

The Unit Vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, and $\boldsymbol{k}$

Two special unit vectors in $\mathbb{R}^2$: $\boldsymbol{i} = \langle 1, 0 \rangle$ and $\boldsymbol{j} = \langle 0, 1 \rangle$, (or in $\mathbb{R}^3$: $\boldsymbol{i} = \langle 1, 0, 0 \rangle$, $\boldsymbol{j} = \langle 0, 1, 0 \rangle$, and $\boldsymbol{k} = \langle 0, 0, 1 \rangle$). They are called basis vectors for $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 6 If $\boldsymbol{u} = 2\,\boldsymbol{i} - 3\,\boldsymbol{j}$ and $\boldsymbol{v} = -5\,\boldsymbol{i} + 7\,\boldsymbol{j}$, find $\boldsymbol{w} = 4\,\boldsymbol{u} - 6\,\boldsymbol{v}$.

Addition of Vectors (the *Resultant*)

Let $\boldsymbol{u} = \langle u_1, u_2 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2 \rangle$, then

$$\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$

Example 7 Two tug boats, *A* and *B*, are are being used to pull a large barge. Ropes tied from the tugs to the barge are separated by an angle of 28°. If tug *A* can pull with a force of 6000 N, and tug B can pull with a force of 4000 N, find the net force on the barge and the direction of the barge relative to tug *B*.

Example 8 Two ropes attached to a ceiling are suspending a 250 pound box. The rope on the left has an angle of depression 70°, and the rope on the right has an angle of depression of 25°. Find the tension on each rope.