11.9 Representing Functions as Power Series

Example 1 Use the sum of a geometric series to find the function represented by $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$, and give its interval of convergence.

Example 2 Find a power series for $f(x) = \frac{x^3}{1-x}$

Example 3 Find the power series for $f(x) = \frac{2x}{1+x^2}$, and write out the first several partial sums.

Integration and Differentiation of Power Series

Theorem

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$ then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R,\ a + R)$ and

(i) $\frac{d}{dx} f(x) = f'(x) = \sum_{n=1}^{\infty} n\, c_n(x-a)^{n-1}$

(ii) $\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} + C$

The radii of convergence for (i) and (ii) are both R, (although convergence at the endpoints may be different.)

Example 4 Find the derivative of the function in *Example 3*.

Example 5 Find a power series for $\ln(x+1)$.

Example 6 Use power series to find $\int \tan^{-1}(x)\, dx$.

Example 7 Use a power series to evaluate $\int_0^{1/2} \frac{1}{1+x^5}\, dx$ to 0.000001

Mathematica Command

Mathematica will calculate a partial sum of a power series of **degree *n*** centered at $x = 0$ as follows:

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Series[2 x / (1 + x^2), {x, 0, 15}]
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$2x - 2x^3 + 2x^5 - 2x^7 + 2x^9 - 2x^{11} + 2x^{13} - 2x^{15} + O[x]^{16}$