11.7 Strategy for Testing Series

In general, there is no set method or set list of tests to apply until one works. It's much more efficient to know what the *form* of the series is in order to test for convergence. Here are some generalities to keep in mind when testing a series:

1. If $\lim_{k\to\infty} a_k \neq 0$ the series diverges.
2. If the terms have a common ratio, or of the form $\sum a\,r^{k-1}$, it's a geometric series and converges if $|\,r\,| < 1$ and diverges if $|\,r\,| \geq 1$.
3. If the series is of the form $\sum \frac{1}{k^p}$ it converges if $p > 1$ and diverges if $p \leq 1$.
4. If the series looks similar to a geometric or *p*-series, a comparison test should be considered. The comparison tests are only good for series with positive terms, so a test for absolute convergence may be necessary.
5. If a series can be written in the form $\sum(a_k - a_{k+1})$ it may be a telescoping series. Find $\lim_{n\to\infty} S_n$.
6. If a series alternates, use the alternating series test; $(a_k \to 0)$. Also can test for Absolute Convergence.
7. If a series contains factorials the Ratio Test is a good option. Note the ratio test does not work for geometric or *p*-series.
8. When a series is of the form $(a_k)^k$ the Root test may be helpful.
9. If $a_k = f(x)$ for each $x = k$, and $f(x)$ is integrable, the Integral Test may be useful.

Summary of Series Tests

Test	Series	Converges when	Diverges when	Note
nth – term	$\sum a_k$		$\lim_{k\to\infty} a_k \neq 0$	Cannot show convergence
Geometric Series	$\sum a\,r^{k-1}$	$\|r\| < 1$	$\|r\| \geq 1$	sum : $S = \frac{a}{1-r}$
p–series	$\sum \frac{1}{k^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum a_k$; and $a_k = f(k) \geq 0$	$\int_1^\infty f(x)\,dx$ converges	$\int_1^\infty f(x)\,dx$ diverges	$R_n = \int_n^\infty f(x)\,dx$
Direct Comparison $(a_n, b_n > 0)$	$\sum a_k$	$0 < a_k \leq b_k$, and $\sum b_k$ converges	$0 < b_k \leq a_k$, and $\sum b_k$ diverges	
Limit Comparison $(a_n, b_n > 0)$	$\sum a_k$	$\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$, and $\sum b_k$ converges	$\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$, and $\sum b_k$ diverges	
Alternating Series	$\sum (-1)^{k-1}\,a_k$	$0 < a_{k+1} \leq a_k$ and $\lim_{k\to\infty} a_k = 0$		$R_n \leq a_{n+1}$
Ratio Test	$\sum a_k$	$\lim_{n\to\infty} \left\|\frac{a_{k+1}}{a_k}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{k+1}}{a_k}\right\| > 1$	Inconclusive if $\lim_{n\to\infty} \left\|\frac{a_{k+1}}{a_k}\right\| = 1$
Root Test	$\sum (a_k)^k$	$\lim_{n\to\infty} \sqrt[k]{\|a_k\|} < 1$	$\lim_{n\to\infty} \sqrt[k]{\|a_k\|} > 1$	Inconclusive if $\lim_{n\to\infty} \sqrt[k]{\|a_k\|} = 1$

Section 11.7 Exercises

(from your text)

1–38 Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n}$

2. $\sum_{n=1}^{\infty} \frac{n - 1}{n^2 + n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n - 1}{n^2 + n}$

5. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

6. $\sum_{n=1}^{\infty} \left(\frac{3n}{1 + 8n}\right)^n$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k + 2)!}$

9. $\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 25}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \sin n$

15. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n + 2)}$

16. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$

17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{k + 5}{5^k}$

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2 + 4n}$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k + 1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j + 5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$

32. $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

33. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

34. $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

35. $\sum_{n=1}^{\infty} \left(\frac{n}{n + 1}\right)^{n^2}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$

38. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

Answers to odd problems are in the back of your text.