11.4 Comparison Test and the Limit Comparison Test

The Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

1. If $0 < a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.

2. If $0 < b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 1 Determine the convergence of $\sum_{k=2}^{\infty} \sqrt{\frac{k}{k^3-1}}$

Example 2 Determine the convergence of $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$

Example 3 Determine the convergence of $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$

The Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.

1. If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 4 Determine the convergence of $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$

Example 5 Determine the convergence of $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Estimating Sums

When using the comparison test, if $a_k \le b_k$,and if b_k is a convergent p-series or geometric series, we can estimate remainder R_n using the n^{th} partial sum of $\sum a_k$ with the remainder T_n from the n^{th} partial sum of $\sum b_k$. If b_k was shown to be convergent using the integral test, we can also estimate the error in a partial sum.

If $a_k \le b_k$ then $s_n = \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} b_n = t_n$, the remainders are $R_n \le T_n$ where $R_n = s - s_n$ and $T_n = t - t_n$.

Example 6 Estimate the value of the series $\sum_{k=1}^{\infty} \frac{2^{k+1}-k}{3^k}$ using the first 10 terms.

Remember, for a geometric series $s = \frac{a}{1-r}$ and $s_n = \frac{a(1-r^n)}{1-r}$.

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Sum[(2^(k+1) - k)/3^k, {k, 1, 10}] // N
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3.18073
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