11.3 The Integral Test and Estimates of Sums

Properties of Convergent Series

1) Let $A = \sum a_k$, then $\sum c\, a_k = c \sum a_k = c\, A$.
2) Let $A = \sum a_k$ and $B = \sum b_k$, then $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A + B$
3) Let $\sum_{k=M}^{\infty} a_k$ be a convergent series, then $\sum_{k=N}^{\infty} a_k$ converges. That is, the addition or subtraction of a finite number of terms does not affect the convergence of a series.

Divergence Test

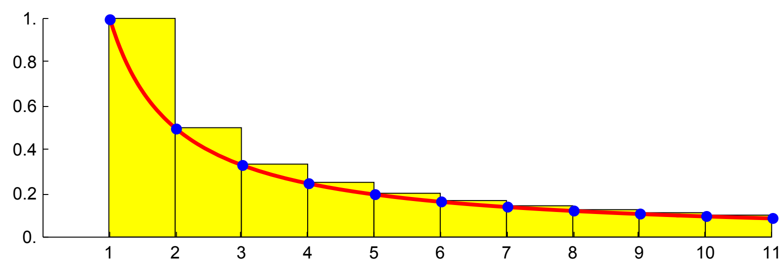
If a series $\sum a_k$ is convergent then $\lim_{k\to\infty} a_k = 0$. Therefore, if $\lim_{k\to\infty} \neq 0$ the series diverges.

Example 1 Determine the convergence of $\sum_{k=1}^{\infty} \frac{2^{k+1}+2^{2k}}{3^k}$

The Harmonic Series

Recall $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{16}}_{>\frac{1}{2}} + \ldots$ is a divergent series even though $\lim_{k\to\infty} \frac{1}{k} = 0$

Another way to show it diverges is to compare the series to the **area** under the function $f(x) = \frac{1}{x}$ from $x = 1$ to $x = \infty$.



$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \int_1^{\infty} \frac{1}{x} dx = \infty$$

The Integral Test

Suppose f is a continuous, positive, decreasing function for $x \geq 1$ and let $a_n = f(n)$ for $n = 1, 2, 3 \ldots$. Then, $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\, dx$ either both converge or both diverge. In the case of convergence, the value of the integral is *not* usually equal to the value of the series.

Example 2 Determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^2}$

The p-series

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Example 3 Determine the convergence of $\sum_{k=4}^{\infty} \frac{5k^3}{7k^\pi}$

Estimating a Series with Positive Terms

Let f be a continuous, positive, decreasing function for $x \geq 1$ and let $a_n = f(n)$ for $n = 1, 2, 3, 4 \ldots$. Also, let $s = \sum_{n=1}^{\infty} a_n$ be convergent and let $s_n = \sum_{k=1}^{n} a_k$. Define the remainder $R_n = s - s_n$, then R_n is bounded as follows:

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx$$

Example 4 Estimate the series $\sum_{k=1}^{\infty} \frac{4}{k^3}$ using the first five terms, and estimate the error in the aproximation. How many terms are needed to approximate the series to within 0.001?

In[128]:= $N\left[\sum_{k=1}^{45} \frac{4}{k^3}, 20\right]$

Out[128]= 4.8072616623660473961

The “actual” value...

In[127]:= $N\left[\sum_{k=1}^{\infty} \frac{4}{k^3}, 20\right]$

Out[127]= 4.8082276126383771416