11.2 Series

The sum of an infinite sequence $\{a_n\}$ is called an **infinite series**, or just a **series**. That is a series is

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 \ldots$$

Example 1 Find $\sum_{i=1}^{\infty} i = 1 + 2 + 3 + 4 \ldots$

Example 2 Find $\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

To determine whether a series has a value, or limit, we can use the idea of limits of *partial sums*.

Given a sequence $\{a_n\}$ or the series $\sum_{i=1}^{\infty} a_n$, a partial sum, s_n, is denoted as

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

If the sequence of partial sums $\{s_n\}$ is convergent, and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent, and the sum s is called the **sum** of the series. Otherwise, the series is called **divergent**.

The Geometric Series

The *geometric series* is a series that can be written as

$$\sum_{i=1}^{\infty} a\,r^{i-1} = a + a\,r + a\,r^2 + a\,r^3 + \ldots + a\,r^{n-1} + \ldots; \quad a \neq 0$$

Example 3 Show that the geometric series converges to $s = \frac{a}{1-r}$ when $|r| < 1$, and divergent if $|r| \geq 1$.

Example 4 Consider the Sequence $a_n = 4\,(0.95)^{n-1}$.

a) Find s_8

b) Find $s = s_\infty$

Example 5 Find the value of the series $s = -9 + \frac{27}{4} - \frac{81}{16} + \frac{243}{64} - \frac{729}{256} \ldots$

Example 6 Evaluate $\sum_{k=1}^{\infty} \frac{4^{k-1}}{5^{k+1}}$

Example 7 Show that the **Harmonic Series:** $s = \sum_{i=1}^{\infty} \frac{1}{i}$ is divergent.

Example 8 Find the value of the *telescoping* series: $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$

Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 9 Show the series $\sum_{i=1}^{\infty} \frac{2i+3}{5i}$ diverges.