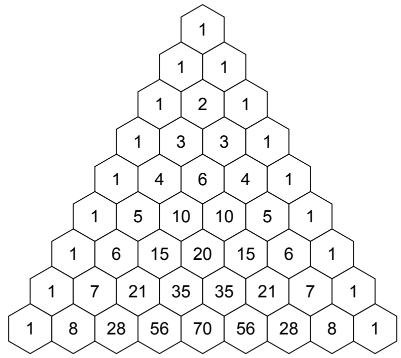
11.11 The Binomial Theorem

Recall Pascal's Triangle and how it can be used to find the coefficients of a binomial raised to a positive integer:



The rows are numbered $k = 0,\ k = 1,\ k = 2 \ldots$, and the elements in each row are numbered $n = 0,\ n = 1,\ n = 2 \ldots$. The first 56 in the bottom row corresponds to $k = 8$ and $n = 3$. Also, when the binomial $(a + b)^8$ is expanded we get

$$a^8 + 8\,a^7\,b + 28\,a^6\,b^2 + 56\,a^5\,b^3 + 70\,a^4\,b^4 + 56\,a^3\,b^5 + 28\,a^2\,b^6 + 8\,a\,b^7 + b^8$$

Notice the coefficients are the values in row $k = 8$. Also, the coefficients can be represented in binomial form $\binom{k}{n}$, i.e., $\binom{8}{3} = 56$. The calculation of the binomial $\binom{k}{n}$ uses factorials, and is given by

$$\binom{k}{n} = \frac{k!}{(k-n)!\,n!}; \quad \text{therefore } \binom{8}{3} = \frac{8!}{5!\cdot 3!} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 56$$

When expanded and simplified, the binomial coefficient can also be written as:

$$\binom{k}{n} = \frac{k(k-1)\,(k-2)\,(k-3)\cdots(k-n+1)}{n!}$$

We can now write the binomial theorem in the compact form:

$$(a + b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}\, b^n$$

If we let $a = 1$ and $b = x$, then we have:

$$(1 + x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

Example 1 Find the Maclaurin series for $f(x) = (1 + x)^k$.

Newton expanded the binomial theorem so that k did not have to be a positive integer, which means we can find the Maclaurin series for $(1 + x)^k$.

The Binomial Series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $\binom{k}{n} = \frac{k(k-1)(k-2)(k-3)\cdots(k-n+1)}{n!}$; converges when $|x| < 1$; diverges when $|x| > 1$; converges at $x = 1$ if $-1 < k \le 0$, and converges at both ± 1 when $k \ge 1$.

Example 2 Expand the power series for $f(x) = \frac{1}{(1+x)^3}$

Example 3 Find a Maclaurin series for $\frac{x^3}{\sqrt{25-x}}$ using a binomial series.