11.10 Taylor and Maclaurin Series

Theorem

If f has a power series expansion centered at a with radius of convergence R, that is

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then the coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The **Taylor Series of the function *f* at *a*** is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \ldots$$

Note: if $a = 0$ the series is called a **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$

However, does this mean that every function can be represented by a power series?

Taylor Polynomials

Let f be a function whose $(n+1)$st derivative $f^{(n+1)}(x)$ exists for all x in an open interval I containing a. A Taylor polynomial of degree n is

$$f(x) \approx T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Therefore, we can write $f(x) = T_n(x) + R_n(x)$ where $R_n(x)$ is the remainder of the Taylor series for $f(x)$. As long as $R_n(x) \to 0$ as $n \to \infty$, then the function f is equal to the sum of its Taylor series, that is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and is convergent for all x in the interval $(a-R,\ a+R)$

Theorem: Convergence of a Taylor Series

If a function f has derivatives of all orders in an open $I = (a-R,\ a+R)$, and if $\lim_{n\to\infty} R_n(x) = 0$ for all x for all x in I, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all x in I.

Example 1 Find the Maclaurin series for $f(x) = e^x$.

Example 2 Find the power series for $f(x) = \sin(x)$ centered at $x = 0$.

Example 3 Differentiate the Power series for $\sin(x)$ to find the series for $\cos(x)$.

Example 4 Find the Taylor polynomial for $f(x) = 2x^3 - 14x + 6$ centered at $x = 3$.

Example 5 Find the first four terms in the Maclaurin series for $e^{-x}\cos(x)$ by multiplying the two series together.

Example 6 Evaluate $\int_0^1 x^2 e^{-x^2}\,dx$ to 4 decimal places.