10.2 Calculus With Parametric Functions

Tangent Lines

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$= \frac{g'(t)}{f'(t)}$$

Example 1 Find the equation of the line tangent to the curve $x = t^2 - 5t$, $y = \ln(t) + 1$ when $t = 1$

Example 2 Find the concavity, or $\frac{d^2y}{dx^2}$, of the above function at $t = 2$ (and possibly the inflection points).

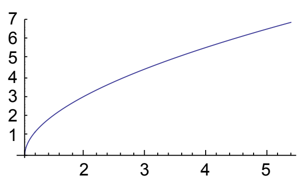
Note: $\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dt}\frac{dt}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Areas

To calculate the area under the function $y = F(x)$ where F is parameterized $F = \{x = f(t),\ y = g(t)\}$ as t transverses from α to β, we can adapt the definite integral as such,

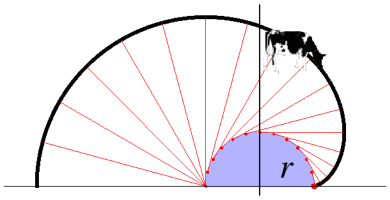
$$\int_a^b F(x)\, dx = \int_a^b y\, dx = \int_\alpha^\beta g(t)\, f'(t)\, dt$$

Example 3 Find the are under the curve $F = \left\{e^t - t,\ 4e^{t/2} - 4\right\}$, for $0 \le t \le 2$



(Can you set up the integral for the area using an explicit function?)

Example 4 A cow is tied to a circular silo with just enough rope to reach the other side. Find the area of grass the cow has to graze. (The figure only shows half of the total area.)



Arc Length

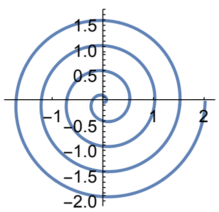
Recall the arc-length formula $L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. We can rewrite this as

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_\alpha^\beta \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt$$

If $\frac{dx}{dt}>0$ we have

$$L=\int_\alpha^\beta \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt=\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Example 5 Find the length of the spiral $\left\{\frac{t}{4\pi}\cos(t),\ \frac{t}{4\pi}\sin(t)\right\}$ for $0\le t\le 8\pi$. You may need an integration table to evaluate the integral. Also, estimate the length using four circles each with an average radius.



Surface Area

Surface area can be derived in a similar way and results in

$$S=\int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_\alpha^\beta 2\pi y\,ds$$

Example 6 Set up the integral to calculate the surface area obtained by revolving $\{x=\sin^2(t),\ y=\sin(3t)\}$ for $0\le t\le \pi/3$ about the x-axis.