How Does Your Calculator Numerically Integrate?

Most calculators use a numerical integration method called Adaptive Gauss-Kronrod Quadrature. Quadrature is a fancy name for numerical integration for definite integrals by evaluating a function $f(x)$ at various points x_i, called nodes, and multiplying the function values by corresponding weights, w_i, and summing the products. The trapezoid rule and Simpson's rule are just two particular quadrature rules.

For instance, if we expand Simpson's Rule we get

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$
$$\approx \frac{\Delta x}{3} f(x_0) + \frac{4\Delta x}{3} f(x_1) + \frac{2\Delta x}{3} f(x_2) + \frac{4\Delta x}{3} f(x_3) + \cdots + \frac{2\Delta x}{3} f(x_{n-2}) + \frac{4\Delta x}{3} f(x_{n-1}) + \frac{\Delta x}{3} f(x_n).$$

The nodes are $x_i = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ and the weights are $w_i = \left\{\frac{\Delta x}{3}, \frac{4\Delta x}{3}, \frac{2\Delta x}{3}, \frac{4\Delta x}{3}, \ldots, \frac{4\Delta x}{3}, \frac{\Delta x}{3}\right\}$.

Gauss-Kronrod Quadrature uses predefined nodes, x_i, and weights, w_i, depending on the desired accuracy. Typically, the first approximation is a GK15-31 approximation, meaning it evaluates the function at 15 points, and at 31 points. If the difference of theses two approximation is small (i.e., less than 0.00000001 or some other specified value) the procedure is ended and the result is displayed. If the difference is too large, the calculator computes a GK31-63 (using the previous 31 points, and computing 32 additional points). This process is repeated until the desired accuracy is achieved.

The nodes and weights are always the same regardless of the function, and are hard-wired into every calculator that is capable of numerical integration. For instance, the nodes and weights for the GK15 rule are:

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<< NumericalDifferentialEquationAnalysis`
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  TableHeadings → {None, {"nodes", "weights"}}], FontFamily → "Arial", 12]
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nodes	weights
−0.987993	0.0307532
−0.937273	0.070366
−0.848207	0.107159
−0.724418	0.139571
−0.570972	0.166269
−0.394151	0.186161
−0.201194	0.198431
0.	0.202578
0.201194	0.198431
0.394151	0.186161
0.570972	0.166269
0.724418	0.139571
0.848207	0.107159
0.937273	0.070366
0.987993	0.0307532

One thing to notice is that the nodes are between $x = -1$ and $x = 1$, and the weights sum to 2. This means every function that is numerically integrated needs to be transformed so that it is defined and constrained to the interval [−1, 1]. So, suppose you are interested in numerically evaluating $\int_4^{10} x^2\,dx$. The integral first needs to be translated so that it is centered at the origin, and then compressed so that it is bounded on [−1, 1]. However, to maintain the area of the original region the function needs to be vertically scaled (stretched) by the same amount as the compression. The following demonstrates the calculations your calculator does during numerical integration:

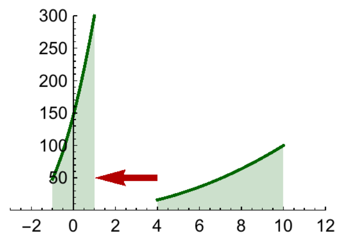
Transforming the integral $\int_4^{10} x^2\, dx$ for a Gauss-Kronrod Quadrature:

Step 1. Translate the interval to the origin. That is, translate using $f(x+k)$, where $k = \frac{b+a}{2}$, i.e, left 7: $(x+7)^2$

Step 2. Compress the function to be defined on the interval [−1, 1] using $f(c\,x+k)$ where $c = \frac{b-a}{2}$, i.e. compress by 3: $(3\,x+7)^2$

Step 3. Vertically scale the function by the same compression amount in step 3: $c\,f(c\,x+k)$. Vertically stretch by 3: $3\,(3\,x+7)^2$. Or in general:

$$g(x) = \frac{b-a}{2} f\left(\frac{b-a}{2} x + \frac{b+a}{2}\right)$$



This transformed function, $g(x)$, is now evaluated at the specified *nodes*, multiplied by the corresponding *weights*, and summed:

$$A \approx \sum_{i=1}^{n} f(x_i)\, w_i\,.$$

To simplify the calculations, we can approximate the above integral using the **GK3** nodes and weights:

	Nodes	Weights
1	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
2	0	$\frac{8}{9}$
3	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

Compare the value against the actual analytic value.

The real power with the Gauss-Kronrod Quadrature is that a GKn method can evaluate a polynomial of degree $2\,n-1$ exactly. This means the GK3 used above can actually evaluate **any** 5th degree polynomial exactly, at only three sample points!! Therefore, a GK31 rule will approximate any 61-degree polynomial exactly with only 31 function values!!! “Holey integration batman!”

Example 1 The natural logarithm function, ln(x), is defined as $\ln(x) = \int_1^x \frac{1}{t}\, dt$. Calculate ln(6) using a GK5 approximation.

The result has an error of about 0.0004. Calculate n for both the trapezoidal rule and Simpson’s rule to attain the same accuracy.

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TableForm[GaussianQuadratureWeights[5, -1, 1],
 TableHeadings → {{1, 2, 3, 4, 5}, {"nodes", "weights"}}]
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	nodes	weights
1	-0.90618	0.236927
2	-0.538469	0.478629
3	0.	0.568889
4	0.538469	0.478629
5	0.90618	0.236927