9.6 First Order Linear Differential Equations - Integrating Factors

A first order linear DE can be written in the form $y' + f(x)\,y = g(x)$. If $g(x) = 0$ the equation is separable. If $g(x) \neq 0$, we need to use an *integrating factor*, $\mu(x)$, to find the solution. First multiply the entire equation by the unknown function $\mu(x)$, and then assume the left-hand side of the equation is the derivative of a product of $\mu(x)$ and $y(x)$

$$y' + f(x)\,y = g(x)$$
$$\mu(x)\,y' + \mu(x)\,f(x)\,y = \mu(x)\,g(x)$$
$$D_x[\mu(x)\,y] = \mu(x)\,g(x)$$
$$\mu(x)\,y' + \mu'(x)\,y = \mu(x)\,g(x)$$

For the second and fourth equation above to be equivalent we must have

$$\mu'(x)\,y = \mu(x)\,f(x)\,y$$

This is a separable equation

$$\mu'(x)\,y = \mu(x)\,f(x)\,y$$
$$\mu'(x) = \mu(x)\,f(x)$$
$$\frac{1}{\mu}\,d\mu = f(x)\,dx$$
$$\int \frac{1}{\mu}\,d\mu = \int f(x)\,dx$$
$$\ln|\mu| = \int f(x)\,dx$$
$$\mu = e^{\int f(x)\,dx}$$

Using the third equation from top we can solve for y by integrating each side and solving for y:

$$D_x[\mu(x)\,y] = \mu(x)\,g(x)$$
$$\mu(x)\,y = \int \mu(x)\,g(x)\,dx$$
$$y = \frac{1}{\mu(x)} \int \mu(x)\,g(x)\,dx$$

where $\mu(x) = e^{\int f(x)\,dx}$.

Example 1 Solve the first order linear differential equation: $y' + 2y = x\,e^{-x}$

Example 2 Find the integrating factor, and then the general solution to the differential equation: $x\,y' + 3y = x + 2$

Example 3 Solve the initial value problem: $y' + x\,y = 3\,x$, $y(0) = 5$.

Example 4 A 2000 gallon tank currently has a concentration of 20 ounces per gallon of ammonia and volume of 1200 gallons. Liquid is being drained at a rate of 5 gallons per minute and a solution of 5 ounces per gallon of ammonia is entering at a rate of 10 gallons per minute. Find the concentration of ammonia in the tank when it becomes full of solution.

Example 5 A ***Bernoulli*** equation is a non-linear equation in the form $y' + f(x)\,y = g(x)\,y^n$ which has known exact solutions (by using the substitution $z = y^{1-n}$ and multiplying by $(1 - n)\,y^{-n}$).

Find the solution to the Bernoulli equation: $y' - \frac{2}{x}\,y = -x^2\,y^2$