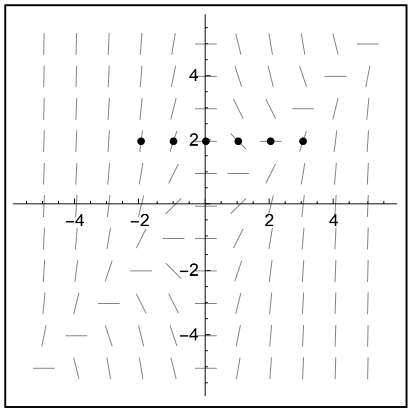
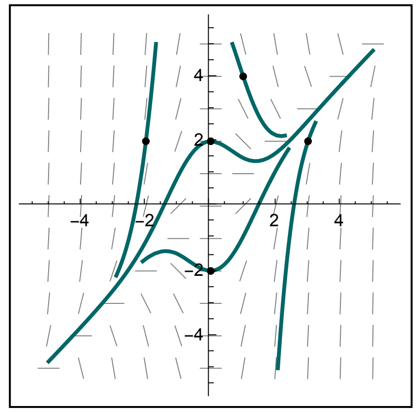
9.2 Direction Fields and Euler's Method

Consider the differential equation $y' + xy = x^2$. Solving for y' allows us to create a **direction field** by calculating the slope (y') at various points (x, y), and plotting short line segments at these points. Solution curves through an initial point can then be estimated. Some sample points and solution curves are given below.

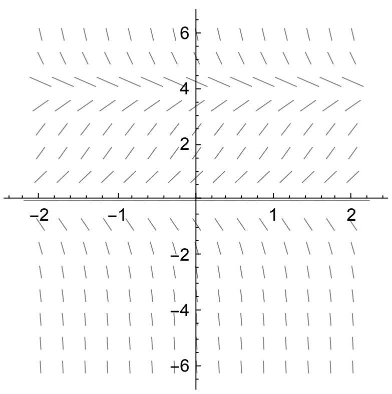
$$y' = x^2 - xy$$

pt	slope
(-2,2)	8
(-1,2)	3
(0,2)	0
(1,2)	-1
(2,2)	0
(3,2)	3





Example 1 The slope field for the *autonomous* (independent variable is missing) differential equation $y' = 4y - y^2$ is given. Plot various solution curves and find the equilibrium solutions.



Numerical Solutions - Euler's Method

A first order differential equation can be written in the form $y' = F(x, y)$, meaning we can approximate the solution curve with a linear function at any point particular (x_0, y_0). For a sufficiently small change in x, i.e. $x_1 = x_0 + \Delta x$, the point on the line (x_1, y_1) will be close to the actual point on the solution curve when $x = x_1$. Using this new point (x_1, y_1) we can estimate (x_2, y_2), etc. This iterative process is called *Euler's Method*.

Since $y' = F(x, y)$, if (x_0, y_0) is a point on the solution curve, then $y_1 = y_0 + h\,F(x_0, y_0)$ where h is the stepsize and $x_1 = x_0 + h$. Hence, $y_2 = y_1 + h\,F(x_1, y_1)$ and $x_2 = x_1 + h$, or more generally

$$y_{i+1} = y_i + h\,F(x_i, y_i)$$
$$x_{i+1} = x_i + h$$

Example 2 Use Euler's method to approximate $y(2)$ for the initial value problem $y' = 2x + y$ where $y(1) = 4$ using a step size $h = 0.1$, and compare it to the exact value $y(2) = 8e - 6$.

Example 3 (a) Use Euler's method with step size 0.2 to estimate $y(1.4)$, where $y(x)$ is the solution of the initial-value problem $y' = x - xy$, $y(1) = 0$. (b) Repeat part (a) with step size 0.1.

A direction field for the differential equation is given below: