5.2 Riemann Sums and The Definite Integral

In section 5.1 we saw that the area under a curve of a continuous positive function on the interval $a \le x \le b$ can be expressed as the limit of a summation:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

(with appropriate values for x_i). The summation on the right is known as a **Riemann Sum**, named after Bernhart Riemann. The limit of the Riemann sum as $n \to \infty$ has another name and simplified notation called a **definite integral**.

Definition of a Definite Integral

A function $f(x)$ defined on the interval $[a, b]$ is said to be *integrable* on $[a, b]$ if $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = \frac{b-a}{n}$, exists. This limit is called the **definite integral of *f* from *a* to *b*** and is written

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Example 1 Geometrically evaluate: $\int_2^5 (2x+1)\,dx$

Example 2 Geometrically evaluate: $\int_0^3 \sqrt{9-x^2}\,dx$

Properties of Definite Integrals

Let *f* and *g* be integrable functions on the interval $[a, b]$, with $a < c < b$, and *k* be a constant.

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Example 3 Let $f(x) = \begin{cases} \sqrt{4-x^2} & \text{if } -2 \le x < 2 \\ -x+2 & \text{if } x \ge 2 \end{cases}$. Evaluate $\int_4^{-2} f(x)\,dx$.

To evaluate some definite integrals using the limit of a Riemann sum, we need some useful summation formulas:

Summation Formulas

1. $\sum_{k=1}^{n} c = c\,n$

2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Example 4 Use the summation formulas to evaluate: $\sum_{k=1}^{10} \left(k^2 - 3k + 4\right)$

Example 5 Use limits and the summation formulas to find the exact value of the area under $f(x) = x^2$ on the interval [0, 4].

Example 6 Use limits and the summation formulas to find the exact value of the area under $f(x) = 2x + 4$ on the interval [1, 3].

Challenge 7 Evaluate $\int_1^2 x^3\,dx$ using the limit of the Riemann Sum.