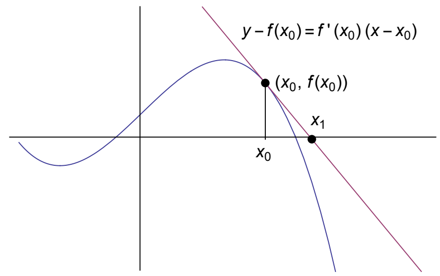
4.9 Newton's Method

Newton's Method is one of many iterative numerical methods to solve an equation in the form $f(x) = 0$. It uses the tangent line at a point $x = x_0$ and obtains a "better" approximation, x_1, which is the x-intercept of the tangent line. The process is repeated to find a second approximation, x_2, and so on until the desired accuracy is reached, or until the method fails.



Using the tangent line to find the x-intercept, x_1, gives:

$$y - f(x_0) = f'(x_0)(x - x_0)$$
$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$-\frac{f(x_0)}{f'(x_0)} = x_1 - x_0$$
$$x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$$

This gives the general rule for Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 1 Find the solution to $-x^3 + 4x = x - 1$ starting with $x_0 = 1.5$.

Using *Mathematica* to find the roots we get:

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FindRoot[-x^3 + 3 x + 1 == 0, {x, 1.5}]
{x → 1.87939}
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Example 2 Where do the graphs of $f(x) = \cos(x)$ and $y = x$ intersect?

Example 3 Find the minimum value of $f(x) = x^4 + 3x^2 - 2x$.

Example 4 Use Newton's Method to approximate $\sqrt[5]{70}$ to 5 decimal places.

Example 5 Find all the solutions to the equation $\ln(x) = x - 2$. Start by drawing a sketch to approximate the initial starting value.