4.3 Derivatives and the Shape of a Graph

Increasing/Decreasing Test

1. A function is **increasing** on an open interval I if $f'(x) > 0$ for all x in I.
2. A function is **decreasing** on an open interval I if $f'(x) < 0$ for all x in I.

Example 1 Determine the intervals the function $f(x) = x^3 + 3x^2 - 24x + 28$ is increasing and decreasing and make a sketch of the function.

The First Derivative Test

Suppose $x = c$ is a critical number of f. The,

1. If $f'(x)$ changes sign from positive to negative at c, then $f(c)$ is a local maximum.
2. If $f'(x)$ changes sign from negative to positive at c, then $f(c)$ is a local minimum.

Example 2 Find and classify the local extrema for the function $f(x) = \frac{x^2-3x+5}{x^2+1}$, and make a rough sketch of the function.

Concavity and the Second Derivative

Consider the four pieces of a function given in *Figure 1*. Notice that a function can be increasing, i.e., $f'(x) > 0$, and increasing more, or increasing but increasing less. This concept is called concavity.

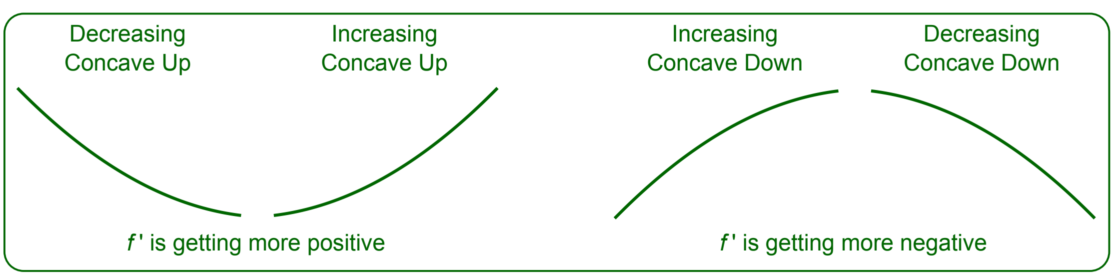

Figure 1: Concavity

Concavity Manipulate

The Second Derivative Test

Suppose f is differentiable for all x in an open interval (a, b) and that there is a critical number c in (a, b) for which $f'(c) = 0$.

1. $f(c)$ is a relative minimum if $f''(c) > 0$.
2. $f(c)$ is a relative maximum if $f''(c) < 0$.

If $f''(c) = 0$ then use the first derivative test to determine whether $f(c)$ is a relative extrema.

Example 3 Use the second derivative test to classify the local extrema for the function $f(x) = x^3 - 3x^2 - 9x + 4$, and find where it changes from concave up to concave down.

Note: If a function f has a point of inflection at c then $f''(c) = 0$ or $f''(c)$ does not exist. However, $f''(c) = 0$ does not guarantee an inflection point.

Example 4 Use the second derivative test to determine the local extrema for $f(x) = x e^{-x^2}$.

Example 5 Make a sketch of the function that satisfies the following conditions:

$f'(1) = f'(-1) = 0$; $f'(x) < 0$ if $|x| < 1$; $f'(x) > 0$ if $1 < |x| < 2$; $f'(x) = -1$ if $|x| > 2$; $f''(x) < 0$ if $-2 < x < 0$; inflection point at (0, 1).