3.5 The Chain Rule

Objectives: Use the chain rule to find the derivative of composite functions; find the derivative of a general exponential function.

Suppose we need to find the derivative of $\left(x^2 - 5x + 1\right)^3$.

The "hard" way:

$$\frac{d}{dx}\left[\left(x^2 - 5x + 1\right)^3\right] = \frac{d}{dx}\left[x^6 - 15x^5 + 78x^4 - 155x^3 + 78x^2 - 15x + 1\right]$$
$$= 6x^5 - 75x^4 + 312x^3 - 465x^2 + 156x - 15$$
$$= \ldots \text{some fancy factoring} \ldots$$
$$= 3(2x - 5)\left(x^2 - 5x + 1\right)^2 \qquad \blacksquare$$

Now, the "easy" way.

The Extended Power Rule

If n is a real number and $f(x)$ is differential, then

$$\frac{d}{dx}\left[f(x)^n\right] = n\left[f(x)^{n-1}\right] \cdot f'(x)$$

Example 1 Find the derivative of: $f(x) = \sqrt{2x^3 - 4x + 1}$

Example 2 Find the derivative of: $f(x) = (2x + 3)^4 (4x - 1)^7$. Simplify by factoring, and find where $f'(x) = 0$.

Example 3 Find the derivative of: $f(x) = \sqrt[3]{\frac{2x+1}{x^2+5}}$

The Chain Rule

If f and g are both differentiable and $h = f \circ g$ is the composite function defined by $h(x) = f(g(x))$, then h is differentiable and h' is given by the product

$$h'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$ are both differentiable then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 4 Find the derivative of: $f(x) = \cos(3x^2 - 5)$

Example 5 Find the derivative of: $f(x) = \tan\left(e^{-3x^2+x}\right)$

Example 6 Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

Derivative of a General Exponential Function

Example 7 Use the properties of logarithms to find the derivative of $f(x) = 5^x$.

Definitions

$\frac{d}{dx}[b^x] =$

$\frac{d}{dx}\left[b^{g(x)}\right] =$

Example 8 Find the equation of the tangent line to the function $f(x) = x\,2^{-x}$ when $x = 1$