2.6 Limits at Infinity; Horizontal Asymptotes

Objectives: Evaluate limits at infinity; identify horizontal asymptotes of functions.

In section 2.2 we looked at infinite limits as $x \to a$ which gave *vertical asymptotes*. It's also important to know the behavior of a function as x gets large, that is, when $x \to \infty$ or $x \to -\infty$.

Example 1 Make a table of values to investigate the value of $f(x) = \frac{\sqrt{4x^2+4x+1}}{3x}$ as $x \to +\infty$ and $x \to -\infty$.

Definition

Let f be a function defined on some interval (a, ∞). Then, $\lim_{x\to+\infty} f(x) = L$ means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. Similarly, if f is defined on $(-\infty, b)$, then we may have $\lim_{x\to-\infty} f(x) = M$. These limits indicate that the function f has *horizontal asymptotes* $y = L$ and/or $y = M$.

Example 2 Find the limits: a) $\lim_{x\to+\infty} \frac{1}{x}$ b) $\lim_{x\to-\infty} \frac{1}{x^3}$.

Theorem

If $r > 0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$. If x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Example 3 Find: $\lim_{x\to-\infty} \frac{3x^2+2x-6}{2x^2-5x+1}$. Justify each step with limit laws.

Example 4 Find: $\lim_{x\to\infty} \frac{2x+4}{\sqrt{3x^2+1}}$.

Example 5 Find: $\lim_{x\to-\infty} \frac{2x+4}{\sqrt{3x^2+1}}$. Also, find the *x* and *y* intercepts of *f* and make a sketch of the function.

Example 6 Find: $\lim_{x\to\infty} \left(\sqrt{4x^2+x} - 2x\right)$.

Example 7 Find: $\lim_{x\to\infty} \left(x^2 - 2x\right)$.

Example 8 Find: $\lim_{x\to-\infty} \left(x + \sqrt{x^2+2x}\right)$

Example 9 Find (a) $\lim_{x\to\infty} \sin(x)$ and (b) $\lim_{x\to\infty} \frac{\sin(x)}{x}$