2.5 Continuity

Objectives: Determine if a function is continuous at a point; classify discontinuities; use the Intermediate Value Theorem.

Laymen's Definition: A function is *continuous* if the entire graph of the function can be made without lifting your pencil.

Real Definition: A function f is **continuous** at $x = a$ if

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$

Example 1 Determine if the function f is continuous at $x = 3$: $f(x) = \begin{cases} \frac{x^2-9}{3-x} & \text{if } x < 3 \\ -6 & \text{if } x = 3 \\ -3x+3 & \text{if } x > 3 \end{cases}$

Example 2 Find the value of a which will make the function f continuous at $x = -2$: $f(x) = \begin{cases} \frac{x^3+8}{x+2} & \text{if } x < -2 \\ a x^2 + 3 & \text{if } x \geq -2 \end{cases}$

Types of Discontinuities

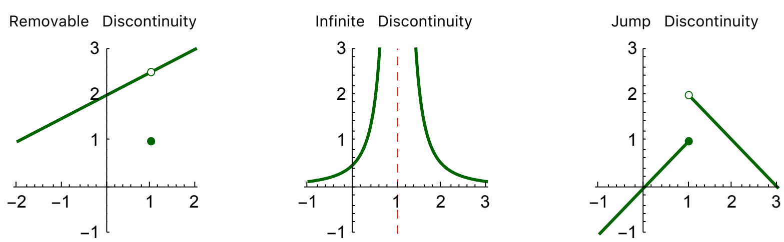

Figure 2.5.2: Discontinuities

Definition

A function f is called **continuous from the right at *a*** if $\lim_{x \to a^+} f(x) = f(a)$, and f is **continuous from the left at *a*** if $\lim_{x \to a^-} f(x) = f(a)$.

Example 3 Show that the function $f(x) = \sqrt{4 - (x-3)^2} + 1$ is continuous on the interval $[1, 5]$.

Theorem

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$ **2.** $f-g$ **3.** cf

4. fg **5.** $\frac{f}{g}$ if $g(a) \neq 0$

This means that **all** *polynomials* are continuous everywhere, that is continuous on $\mathbb{R} = (-\infty, +\infty)$, and all rational functions are continuous on their domain. Most of our familiar functions are continuous on their domain including:

polynomial functions, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions

Example 4 Where is the function $f(x) = \frac{\sqrt{x} + \cos^{-1}\left(\frac{x}{5}\right)}{x-2}$ continuous?

Theorem: Continuity of Composite Functions

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a. Basically, "a continuous function of a continuous function is a continuous function". ☺

Intermediate Value Theorem

An important and useful property of a continuous function is the **Intermediate Value Theorem**

Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in $[a, b]$ such that $f(c) = N$.

Example 5 Show that there is a zero of the function $f(x) = \ln(x) + x^2 - 5x$ on the interval $(4, 5)$.

Example 6 Given the function $f(x) = 2\sec(x)$, find $f(1)$ and $f(2)$. Is there a zero between 1 and 2?