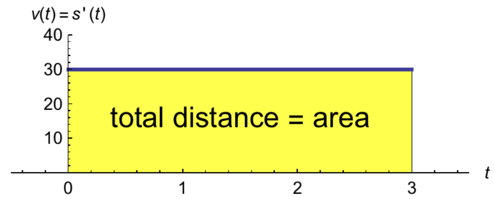
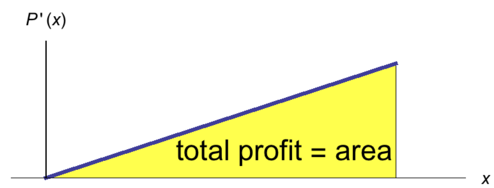
4.2 Definite Integrals and Areas

Consider a car traveling 30 mph for 3 hours, totaling traveling distance is $d = 30\,m/h \times 3\,\text{hr} = 90\,m$. Graphically we can show this:



The total distance traveled is the **area** under the velocity curve, or $s'(t)$.

Similarly, let $P'(x)$ be the marginal profit of a company. The area under the marginal profit function is equal to the total profit.



The Definite Integral

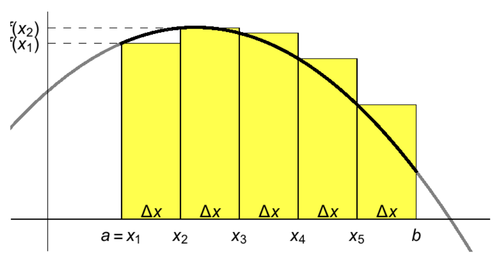
The area under a function $f(x)$ on the interval $[a, b]$ is represented by the **definite integral**:

$$\text{Area} = \int_a^b f(x)\,dx$$

Example 7 Find the area represented by $\int_1^3 (2x+1)\,dx$ geometrically.

Riemann Sums

Non-geometrical areas under the curve of $f(x)$ on the interval $[a, b]$ can be approximated by a **Riemann Sum** using n rectangles whose heights are calculated using $f(x)$ and the width of each rectangle is $\Delta x = \frac{b-a}{n}$



The area under the curve can be approximated by the area of the rectangles

$$\begin{aligned}\text{Area} &\approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x \\ &= (f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5))\,\Delta x \\ &= \sum_{i=1}^{5} f(x_i)\,\Delta x\end{aligned}$$

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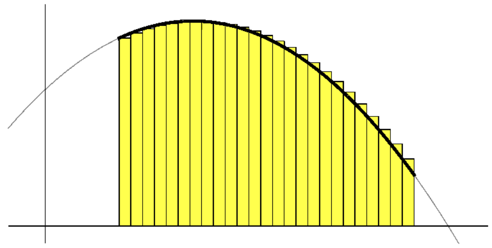
The approximation can be made better by using more rectangles.



Example 2 Approximate the area under the curve $f(x) = x^2 + 1$ on the interval [1, 4] using $n = 5$ rectangles and left-hand endpoints.

Example 3 Approximate $\int_0^2 x\, e^x\, dx$ using a Riemann sum with $n = 4$ and the center-point method.

The **exact** area can be found by taking the limit as $n \to \infty$, or equivalently, as $\Delta x \to 0$.

Definite Integral

Let $y = f(x)$ be a continuous non-negative function over the interval $[a, b]$. The area under curve can be calculated using a Riemann Sum and taking the limit as $\Delta x \to 0$. The result is the **definite integral**.

$$\text{Exact Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

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