3.5 Derivatives of General Exponential and Logarithmic Functions

The Derivative of $f(x) = b^x$

$$\frac{d}{dx}[b^x] = b^x \cdot \ln(b)$$

Proof:

$$\begin{aligned} y &= b^x \\ \ln(y) &= x\ln(b) \\ \frac{1}{y}\cdot y' &= \ln(b) \\ y' &= y\ln(b) \\ y' &= b^x \cdot \ln(b) \quad \blacksquare \end{aligned}$$

Example 1 Find the derivative of $f(x) = 5000 \cdot 1.08^{x/2}$

Example 2 Find the critical numbers for $f(x) = x \cdot 5^{1-x^2}$

The Derivative of $f(x) = \log_b(x)$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x} \cdot \frac{1}{\ln(b)}$$

Derivation:

Example 3 Find the equation of the line tangent to the graph of $f(x) = \log_2\left(x^2 + 4\right)$ when $x = 2$

Example 4 Differentiate $f(x) = 5^{x+1}\log_5(x + 1)$

Logarithmic Differentiation

Consider the function $y = x^x$. This is neither a power function nor an exponential function. Find y' by **first** taking the natural log of each side, differentiate implicitly, and then solve for y' explicitly.

$$\begin{aligned} y &= x^x \\ \ln(y) &= \ln(x^x) \\ \ln(y) &= x\ln(x) \\ \tfrac{d}{dx}[\ln(y)] &= \tfrac{d}{dx}[x\ln(x)] \\ \tfrac{1}{y}\cdot y' &= \ln(x) + \tfrac{x}{x} \\ y' &= y(\ln(x) + 1) \\ y' &= x^x(\ln(x) + 1) \qquad \blacksquare \end{aligned}$$

Example 5 Find the derivative of $y = x^{\ln(x)}$.