8.5 The Dot Product

Definition of the Dot Product

Let $\mathbf{u} = \langle a_1,\ b_1 \rangle$ and $\mathbf{v} = \langle a_2,\ b_2 \rangle$, the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is:

$$\mathbf{u} \cdot \mathbf{v} = a_1\, a_2 + b_1\, b_2$$

Example 1 Find the dot product of the following vectors:

a) $\mathbf{u} = \langle -2,\ 3 \rangle$ and $\mathbf{v} = \langle -2,\ 5 \rangle$

b) $\mathbf{u} = 2\,\mathbf{i} - 5\,\mathbf{j}$ and $\mathbf{v} = -7\,\mathbf{i} + \mathbf{j}$

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\,\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $\| \mathbf{u} \|^2 = \mathbf{u} \cdot \mathbf{u}$

The Dot Product Theorem

If θ is the angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \|\ \| \mathbf{v} \| \cos(\theta) \quad \text{or} \quad \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

Example 2 Use the Law of Cosines to prove the Dot Product Theorem.

Example 3 Find the angle between the vectors $\mathbf{u} = \langle 5,\ 2 \rangle$ and $\mathbf{v} = \langle 5,\ 6 \rangle$.

Example 4 Find the angle between the vectors $\mathbf{u} = \langle -12,\ 6 \rangle$ and $\mathbf{v} = \langle 4,\ 8 \rangle$.

Orthogonal Vectors

Two nonzero vectors **u** and **v** are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 5 Determine c for the vectors to be orthogonal: $\boldsymbol{u} = 8\,\mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{v} = c\,\mathbf{i} + 5\,\mathbf{j}$.

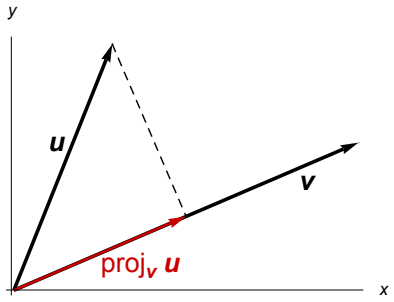
Work

Work is defined as force times distance, or the amount of force f required to move an object a distance d: $w = f \times d$. Suppose a force vector $\boldsymbol{F}$ moves an object along the displacement vector $\boldsymbol{D}$. If θ is the angle separating the two forces, show the work done by $\boldsymbol{F}$ on $\boldsymbol{D}$ is $W = \boldsymbol{F} \cdot \boldsymbol{D}$

Example 6 A force of 50 Newtons in the direction of $\langle 2, 5 \rangle$ moves an object from point $a(-3, 2)$ to $b(8, 4)$. Find the work done.

Projections

An **orthogonal** projection of a vector **u** onto a vector **v** can be thought of the "shadow" of vector **u** made by a light shining overhead:



Scalar Projection: $\text{comp}_{\boldsymbol{v}}\,\boldsymbol{u} =$

Vector Projection: $\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} =$

Example 7 Find the projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ where $\boldsymbol{u} = 5\,\boldsymbol{i} + 8\,\boldsymbol{j}$ and $\boldsymbol{v} = 10\,\boldsymbol{i} + 3\,\boldsymbol{j}$. Resolve $\boldsymbol{u}$ into two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that $\boldsymbol{u}_1$ is parallel to $\boldsymbol{v}$, $\boldsymbol{u}_2$ is orthogonal to $\boldsymbol{v}$, and $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$