8.3 Complex Numbers in Polar Coordinates

Complex Number in the Complex Plane

The complex plane is a rectangular coordinate system where the *x*-axis is the real number line (*re*), and the *y*-axis is the complex number line (*im*). Complex numbers, $a + b\,i$ are plotted in the complex plane, using the coordinates a and b as coordinates (a, b).

Polar Form of a Complex Number

Using trigonometry, the complex number $z = a + b\,i$ can be written in polar form:

$$z = r(\cos(\theta) + i\sin(\theta)) \qquad \text{where } r = \sqrt{a^2 + b^2} \text{ and } a = r\cos(\theta),\quad b = r\sin(\theta)$$

This is sometimes abbreviated as $z = r\,c\,i\,s\,\theta$. Note: r is called the **modulus** and the notation is $r = \|a + b\,i\|$; the angle is called the **argument**, and is found using $\tan(\theta) = \frac{b}{a}$.

Example 1 Convert the polar complex numbers to rectangular form. Approximate to two decimals if necessary.

(a) $5(\cos(30°) + i\sin(30°))$ (b) $8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ (c) $3\,c\,i\,s\,232°$

Example 2 Convert the complex numbers from rectangular form to polar form.

(a) $4 + 2\,i$ (b) $-2 + 3\,i$ (c) -4

The Product and Quotient of Two Complex Numbers

The product of two complex numbers $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$ is

$$z_1 \cdot z_2 =$$

Example 3 Find the product of the given complex numbers. Check your answer converting both numbers to rectangular form and then finding the product.

$z_1 = 4(\cos(45°) + i\sin(45°))$ and $z_2 = 3(\cos(60°) + i\sin(60°))$

$$\frac{z_1}{z_2} =$$

Complex Numbers Raised to a Power

The polar form of a complex number makes it easy to find the value of a complex number raised to a positive integer power, i.e., z^{12}. Using repeated multiplication and the *product* formula gives the following:

DeMoivre's Formula for Powers

Let $z = r(\cos(\theta) + i\sin(\theta))$, then z raised to the n^{th} power is given by

$$z^n =$$

Example 4 Given $z = 2(\cos(36°) + i\sin(36°))$, find z^5. Write answer in rectangular form.

Example 5 Find $(3 + 3i)^7$. Write answer in rectangular form.

The Root of a Complex Number

What is the square-root of 25? What is the 6^{th} root of 64? What is $\left(1 + i\sqrt{3}\right)^6$? What is $\left(1 - i\sqrt{3}\right)^6$. How many 6^{th} roots of 64 do you think there are?

DeMoivre's Formula for Powers

Let $z = r(\cos(\theta) + i\sin(\theta))$, then the n n^{th} roots of z are given by

$$z^{1/n} =$$

Example 6 Find the five 5^{th} roots of -32. Write the roots in exact polar form and rectangular form rounded to three decimal places.

Example 7 Find the four 4^{th} roots of i.