10.5 Rotation of Axes

Recall the general form of a conic is given by

$$A\,x^2 + C\,y^2 + D\,x + E\,y + F = 0$$

Missing from that expression is the term $B\,x\,y$.

$$A\,x^2 + B\,x\,y + C\,y^2 + D\,x + E\,y + F = 0$$

The Angle of Rotation

The affect of the *xy* term is a rotation (and scaling) of the conic, and can also effect the shape. The rotation angle of the conic is found by solving for the acute angle θ in the equation:

$$\cot(2\,\theta) = \frac{A-C}{B}$$

Rotated Coordinate System

Using a new rotated coordinate system $(X,\ Y)$, we can remove the rotation *xy*-term. Equations to convert from $(x,\ y)$ to $(X,\ Y)$ are:

$$x = X\cos(\theta) - Y\sin(\theta) \qquad \text{and} \qquad X = x\cos(\theta) + y\sin(\theta)$$
$$y = X\sin(\theta) + Y\cos(\theta) \qquad \qquad \qquad Y = -x\sin(\theta) + y\cos(\theta)$$

The Discriminant

The *xy* term in the general conic equation also affects the type of conic. To identify the type conic we use the discriminant $B^2 - 4\,A\,C$.

1. If $B^2 - 4\,A\,C = 0$ the graph is a **parabola**.
2. If $B^2 - 4\,A\,C < 0$ the graph is an **ellipse**.
3. If $B^2 - 4\,A\,C > 0$ the graph is a **hyperbola**.

Example 1 Identify the conic and angle of rotation for: $4\,x^2 + 6\,x\,y + 5\,y^2 - 8\,x + 7\,y - 20 = 0$. (To graph on your calculator you would need to solve for *y*)

Example 2 Identify and sketch the curve $73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0$. To find exact values for $\sin(\theta)$ and $\cos(\theta)$ half-angle formulas are useful. Find the *XY*-coordinates of the foci, and then find the *xy*-coordinates of the foci.