3.6 Properties of Rational Functions

Objectives: Find the domain of a rational function; find vertical, horizontal, and oblique asymptotes of rational functions.

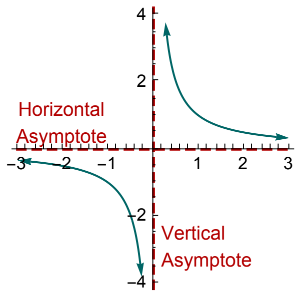
A reduced **rational function** is a quotient of two polynomial functions $p(x)$ and $q(x)$, where p and q have no common factors

$$R(x) = \frac{p(x)}{q(x)}$$

The domain of a rational function is all real numbers except for those which the denominator q is 0.

Example 1 Find the domain of $R(x) = \frac{x^2+3x-4}{x^2-4}$.

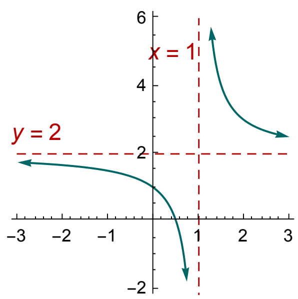
The simplest rational function is the toolkit function $f(x) = \frac{1}{x}$. As $x \to \infty$ $f(x) \to 0$, and as $x \to -\infty$ again $f(x) \to 0$. The horizontal axis is called a **horizontal asymptote**. Notice also that as $x \to 0$ $f(x) \to \pm\infty$. The y axis is a vertical asymptote.



If we translate f to the right 1 and up 2 we get

$$\begin{aligned} f(x-1)+2 &= \frac{1}{x-1} + 2 \\ &= \frac{1}{x-1} + \frac{2x-2}{x-1} \\ &= \frac{2x-1}{x-1} \end{aligned}$$

Therefore, the graph of $y = \frac{2x-1}{x-1}$ is



Example 2 Find the transformations on the toolkit function $f(x) = \frac{1}{x}$ to obtain $R(x) = \frac{3x+5}{x+2}$ and graph.

A few things to notice from example 2:

1. The x intercept is found by finding the zeros of the numerator, $\left(\frac{-5}{3}, 0\right)$.
2. The y intercept is found by evaluating the function at $x = 0$, $\left(0, \frac{5}{2}\right)$.
3. The vertical asymptote at $x = -2$ is found by finding the zeros of the denominator.
4. The horizontal asymptote is now at $y = 3$. Finding the horizontal asymptote (without knowing the transformations first) is a little more complicated.

Finding Horizontal Asymptotes

Let f be a rational function with numerator of degree m and denominator of degree n:

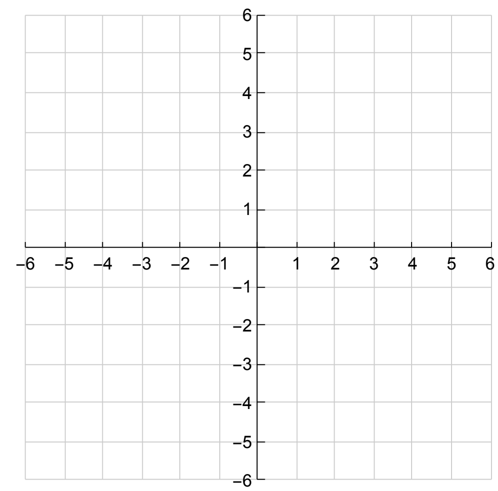
$$f(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}$$

I. If $m < n$ the x axis or $y = 0$ is the horizontal asymptote.

II. If $m = n$ then $y = \frac{a_m}{b_n}$ is the horizontal asymptote.

III. If $m > n$ then there is no horizontal asymptote. Instead, use long division to find a *slant* asymptote.

Example 3 Graph the function $f(x) = \frac{3x^2 - 9x - 12}{2x^2 - 8}$ by finding the asymptotes, intercepts, and additional points if necessary.



Example 4 Find the asymptotes and intercepts for the function $f(x) = \frac{x^2 + 3x + 2}{2x - 2}$ whose graph is given.