3.3 Real Zeros of Polynomials

Objectives: List all possible rational zeros; Use Descartes' rule of signs; Find upper and lower bounds to real zeros.

Rational Zeros Theorem

Let c be a zero of the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$. Also, let p be a factor of a_0 and q be a factor of a_n, then c is a rational number in the form $c = \frac{p}{q}$, meaning if c is a rational zero of a polynomial $P(x)$, the numerator has to be a factor of the constant a_0, and the denominator has to be a factor of the leading coefficient a_n.

Example 1 List all *possible* rational zeros for the function $f(x) = 3x^4 + 2x^3 - 25x^2 - 28x + 12$.

Example 2 Use synthetic division and the remainder theorem to find the rational zeros for the function in example (1).

Example 3 Find the real zeros of $f(x) = 2x^4 - x^3 - 17x^2 + x + 15$.

Descartes' Rule of Signs

Let P be a polynomial with real coefficients.

1. The number of positive real zeros is either equal to the number of sign changes in $P(x)$ or is less than that by an even whole number.

2. The number of negative real zeros is either equal to the number of sign changes in $P(-x)$ or is less than that by an even whole number.

Example 4 Use Descartes' Rule of Signs to determine the possible number of positive and negative real zeros for the polynomial $P(x) = 3x^5 + 16x^4 + 8x^3 - 62x^2 - 75x - 18$

Upper and Lower Bounds Theorem

Let $P(x)$ be a polynomial with real coefficients, and divisor $x - c$, i.e., $\frac{P(x)}{x-c}$:

1. If $c > 0$ and the bottom row of the synthetic division has no negative values, then c is an **upper bound** for the real zeros of P.

2. If $c < 0$ and the bottom row of the synthetic division has values with alternating signs, then c is a **lower bound** for the real zeros of P.

Example 5 Show that the real zeros of the polynomial $P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8$ are in the interval [−3, 5].

Example 6 Find all rational and irrational zeroes of the polynomial. Use the *Rational Zeros Theorem*, the *Upper and Lower Bounds Theorem*, *Descartes' Rule of Signs*, the quadratic formula, and any other factoring techniques that may be useful.

$$P(x) = 8x^5 - 14x^4 - 22x^3 + 57x^2 - 35x + 6$$