3.1 Polynomial Functions and Their Graphs

Objectives: Identify the degree of a polynomial function; identify the zeros and multiplicities for a polynomial; analyze the graph of a polynomial.

A polynomial of degree n is any function that can be written as a sum of terms involving a numerical *coefficient* multiplied by the variable raised to a positive integer. A general polynomial is expressed as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

Examples are: $f(x) = 3x^5 - 2x^4 + 6x^3 - 4x + 1$, $g(x) = 3x^2 + 4x + 3$. The following are **not** polynomials: $F(x) = \frac{3x}{x+1}$, $G(x) = \sqrt{x^2 + 1}$, $H(x) = 2^x$.

The graph of a polynomial is always **smooth** and **continuous**, no holes, gaps, asymptotes, or sharp corners. Figure 1 shows a typical polynomial; figure 2 is not a polynomial.

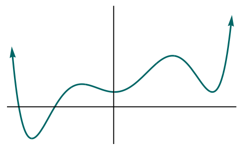


Figure 1

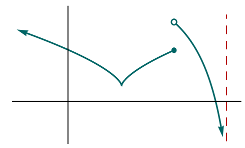


Figure 2

Graphs of $f(x) = x^n$; and End Behavior

Notice the different characteristics of $f(x) = x^n$ depending on whether n is even or odd.

n is Even

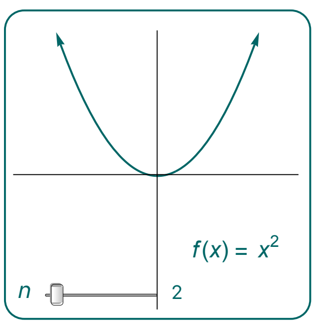


Figure 3: Even Powers

n is Odd

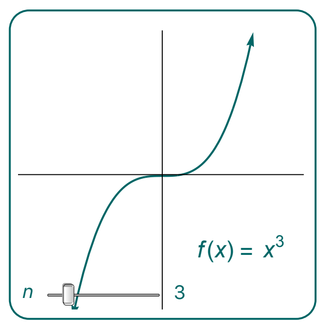
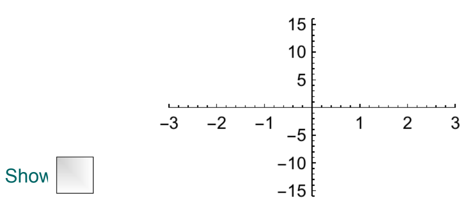
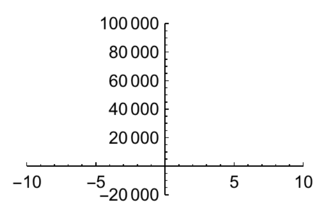


Figure 4: Odd Powers

Take special note of the end behavior (i.e., when $x \to -\infty$ or $x \to \infty$), the shape of the graph, and how the function crosses the x axis.

Example 1 Graph both $f(x) = x^6$ and $g(x) = x^6 - 4x^4 - 3x^2 - 7x + 8$. Zoom out incrementally until you have XRange[-10,10] and YRange[-20000,100000]. What do you notice?





Example 2 Make a sketch of the graph of $f(x) = (x - 4)^5$

Zeros and Multiplicities

Consider the graph of the polynomial $P(x) = \frac{1}{2}(x+3)^2(x+2)(x-1)^3$ given below. The **zeros** of the polynomial are the values of x such that $f(x) = 0$. For this polynomial, the factors $(x+3)$, $(x+2)$, and $(x-1)$ give the zeros $\{-3, -2, 1\}$. The *power* of each binomial factor is the **multiplicity** for that zero. In this case, the multiplicities are 2, 1, and 3. Adding the multiplicities up gives the degree of the entire polynomial $n = 6$. Note, the leading coefficient, $\frac{1}{2}$, is a vertical scaling factor and does not affect the zeros.

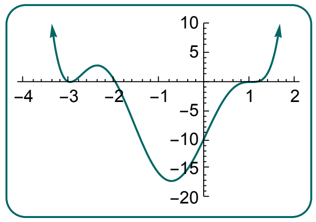


Figure 5

This leads to the following equivalent statements:

1. c is a zero of P.
2. $x = c$ is a solution to $P(x) = 0$.
3. $(x - c)$ is a factor of $P(x)$.
4. $x = c$ is an x-intercept of the graph of P.

Using zeros and multiplicities to graph polynomial functions.

Example 3 Make a sketch of the polynomial $P(x) = 2(x+3)^2(x^2-1)(x-2)$ including the y-intercept.

Example 4 Use the graph of $f(x)$ to completely factor the polynomial $f(x) = 2x^5 - 5x^4 - 27x^3 + 92x^2 - 44x - 48$.

Intermediate Value Theorem

Let $f(x)$ be a continuous function on the interval (a, b). If $a < b$, and $f(a)$ and $f(b)$ are opposite sign, then f has at least one zero between a and b.

Example 5 Use the intermediate value theorem to show there is a zero between 1 and 2 for the function $f(x) = 3x^3 + 10x^2 - 7x - 30$.