2.8 One to One Functions and Inverse Functions

Recall that a function maps a number x in the domain to a number y in the range, $f(x)=y$. An inverse function reverses this mapping, in that it takes a number from the range and maps it back to the domain. For example, if $f(3)=8$ then the inverse function, say g, would be such that $g(8)=3$. The notation for the inverse function of f is f^{-1}, so if $f(3)=8$, then $f^{-1}(8)=3$.

Now, suppose we have a function that maps $f(5)=3$ and $f(2)=3$. The inverse is **not** a function since we would need $f^{-1}(3)=5$ and $f^{-1}(3)=2$. In order for a function f to have an inverse function f^{-1}, f must be a **one-to-one** function.

Consider the function shown below, f maps each value x to a value y, while f^{-1} maps each value in reverse. Notice how the domain and range of are interchanged for f^{-1}; the domain of f becomes the range of f^{-1}, and the range of f becomes the domain for f^{-1}.

$f(x)$ $\Longrightarrow$

x	y
1	4
2	6
3	5
4	7
y	x

$\Longleftarrow$ $f^{-1}(x)$

Inverse Functions

One to One Functions

A function is **one-to-one** if for every x in the domain there is exactly one value y in the range, and for each y in the range there is exactly one value in the domain.

Example 1 Which of the following are one to one functions?

(a) $f(x)=2x-5$ (b) $f(x)=x^2$ (c) $f(x)=\sqrt{x}$ (d) $f(x)=\frac{1}{x}$ (e) $x^2+y^2=9$

Example 2 Show by using numerical values that the given functions *could* be inverses of each other: $f(x)=\frac{2}{3}x-4$, and $g(x)=\frac{3x+12}{2}$.

Steps to Find an Inverse Function

0. Make sure the function is one-to-one, restricting the domain if necessary.
1. Replace $f(x)$ with y.
2. Interchange the x's and y's.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

Example 3 Show that the functions in example (2) are inverses by finding the inverse of f.

Example 4 Find the inverse of $f(x) = \frac{2x+3}{x-6}$.

Characteristics of Inverse Functions

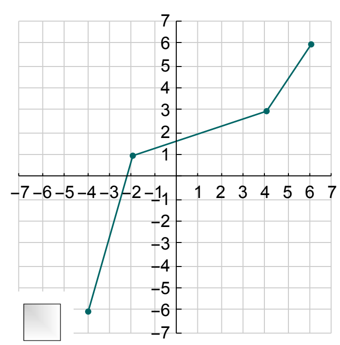
Since an inverse function always returns the original value a function was evaluated at, we have the following relationship for inverse functions:

$$\left(f \circ f^{-1}\right)(x) = x \quad \text{and} \quad \left(f^{-1} \circ f\right)(x) = x$$

Example 5 Find $\left(f \circ f^{-1}\right)(x)$ for the functions in example (2).

Another characteristic of inverse functions is their symmetry around the line $y = x$.

Example 6 Graph the inverse of the function shown.



Graph the Inverse Function

Example 7 Find the inverse of the function $f(x) = x^2 - 8x + 17$ valid for the point (6, 5).