2.7 Combining Functions

The Sum, Difference, Product, and Quotient of Functions

Let \( f \) and \( g \) be functions with domains \( A \) and \( B \). The, \( f + g \), \( f - g \), \( f \cdot g \), and \( f/g \) are defined as follows:

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<tr>
<td>1.</td>
<td>((f + g)(x) = f(x) + g(x))</td>
<td>Domain ( A \cap B )</td>
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<td>2.</td>
<td>((f - g)(x) = f(x) - g(x))</td>
<td>Domain ( A \cap B )</td>
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<td>3.</td>
<td>((f \cdot g)(x) = f(x) \cdot g(x))</td>
<td>Domain ( A \cap B )</td>
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<td>4.</td>
<td>((f/g)(x) = \frac{f(x)}{g(x)})</td>
<td>Domain ( {x \in A \cap B \mid g(x) \neq 0} )</td>
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**Example 1**
Given \( f(x) = \sqrt{x + 5} \) and \( g(x) = x - 2 \) make a sketch of \((f + g)(x)\).

**Example 2**
For \( f(x) = x^2 - 9 \) and \( g(x) = \frac{x+3}{x-2} \) simplify \((f/g)(x)\) and find the domain. Verify by graphing \( f/g \).

Composite Functions

A composite function is when one function, \( f \), is evaluated with another function, \( g \), i.e., \( f(g(x)) \). The notation for a composite function is \((f \circ g)(x) = f(g(x))\).

**Example 3**
Given \( f(x) = x^2 + 2x - 1 \) and \( g(x) = 2x - 3 \) find and simplify \((f \circ g)(x)\). Evaluate \( f(g(4)) \) two different ways.

**Example 4**
Given the graphs of \( f \) and \( g \), find the following expressions:

- (a) \((f \circ g)(2)\)
- (b) \((f \circ g)(-4)\)
- (c) \((g \circ f)(-2)\)
- (d) \((f \circ f)(-8)\)
- (e) \((g \circ g)(3)\)

![Graphs of f and g](image.png)
Example 5  For Example 4 above, find \( f \) and \( g \), and
(a) find \((f \circ g)(x)\) and use it to verify parts (a) and (b) above
(b) find \((g \circ g)(x)\) and use it to verify part (e) above.

Domain of a Composite Function

For a composite function \( f(g(x)) \), since the function \( g \) is evaluated first, the domain is initially restricted to that of \( g \). In addition, since the output of \( g \) is used as input into \( f \) we need to make sure \( g \)'s output does not give values that are restriction on the input for \( f \).

For example, let \( f(x) = \sqrt{x + 3} \) and \( g(x) = \frac{2}{x-4} \). We can see that for \( g \), \( x \neq 4 \). Also, for \( f \) we have the input must be \( x \geq -3 \). Therefore, the output of \( g \) must be greater than or equal to \(-3\), or \( \frac{2}{x-4} \geq -3 \). Solving this inequality we have the domain for \((f \circ g)(x)\) as \((-\infty, \frac{10}{3}] \cup (4, \infty)\).

Example 6  Verify the domain of \( f \circ g \) by graphing \((f \circ g)(x)\) where \( f(x) = \sqrt{x - 1} \) and \( g(x) = \frac{5}{x-4} \).

Example 7  Find the domain for \( f \circ g \) given \( f(x) = \frac{2}{x-3} \) and \( g(x) = x^2 + 4x + 6 \).

Example 8  Find the functions \( f, g, \) and \( h \), such that \( f(g(h(x))) = \sqrt{(2x + 3)^3} + 5 \).